Insight on the parameterized complexity of scheduling problems

Designing FPT algorithms with dynamic programming

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may 17, 2024

New challenges in scheduling Theory, Aussois 2024

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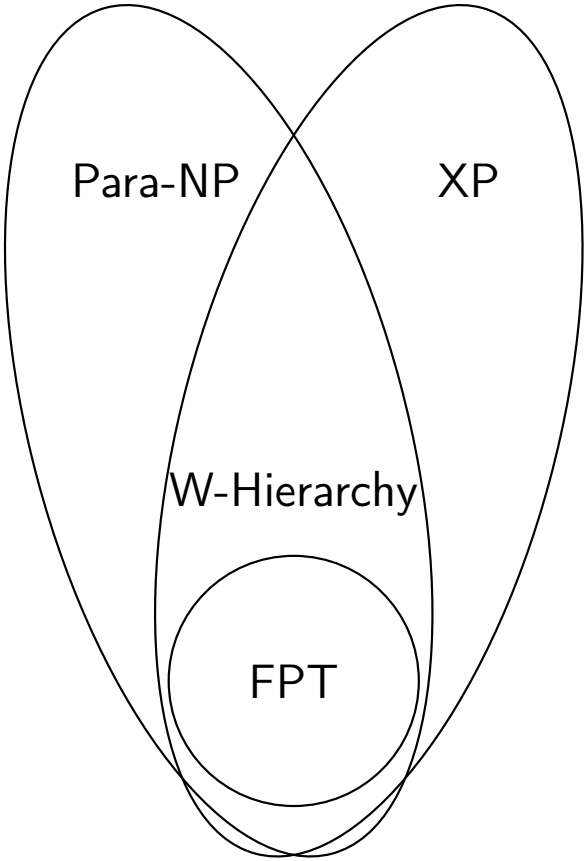
Parameterized complexity classes



A parameterized problem of size n with parameter k :

Definition

FPT is the class of problems solvable by a fixed-parameter tractable algorithm with time complexity $\mathcal{O}(f(k) \times \text{poly}(n))$, where f is a computable function and poly(n) a polynome of n.

Challenges for scheduling problems

- Which problems can be shown FPT?
- Which relevant structural parameters for scheduling?
- How to design FPT algorithms?

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- Which relevant structural parameters for scheduling?
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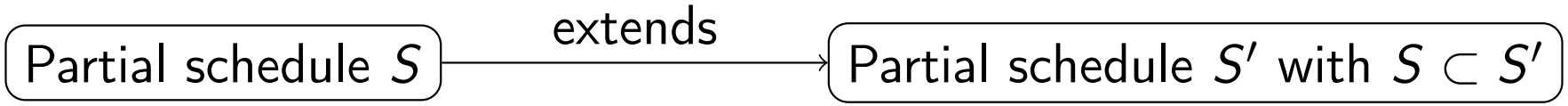
Two main techniques:

1. ILP with special structure (for example Hermelin, Karhi, Pinedo,and Shabtay, 2021)
2. Dynamic programming

Dynamic programming for scheduling



- The second partial schedule extends the first one.
- A state records the minimum information needed from S to build feasible extensions S'
- Dynamic programming algorithm starts from an empty schedule and builds a state graph.

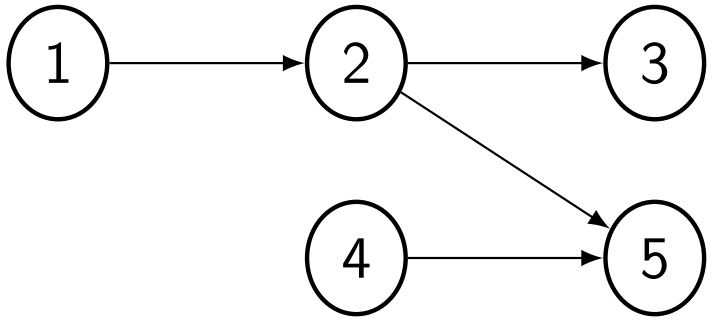
Key arguments

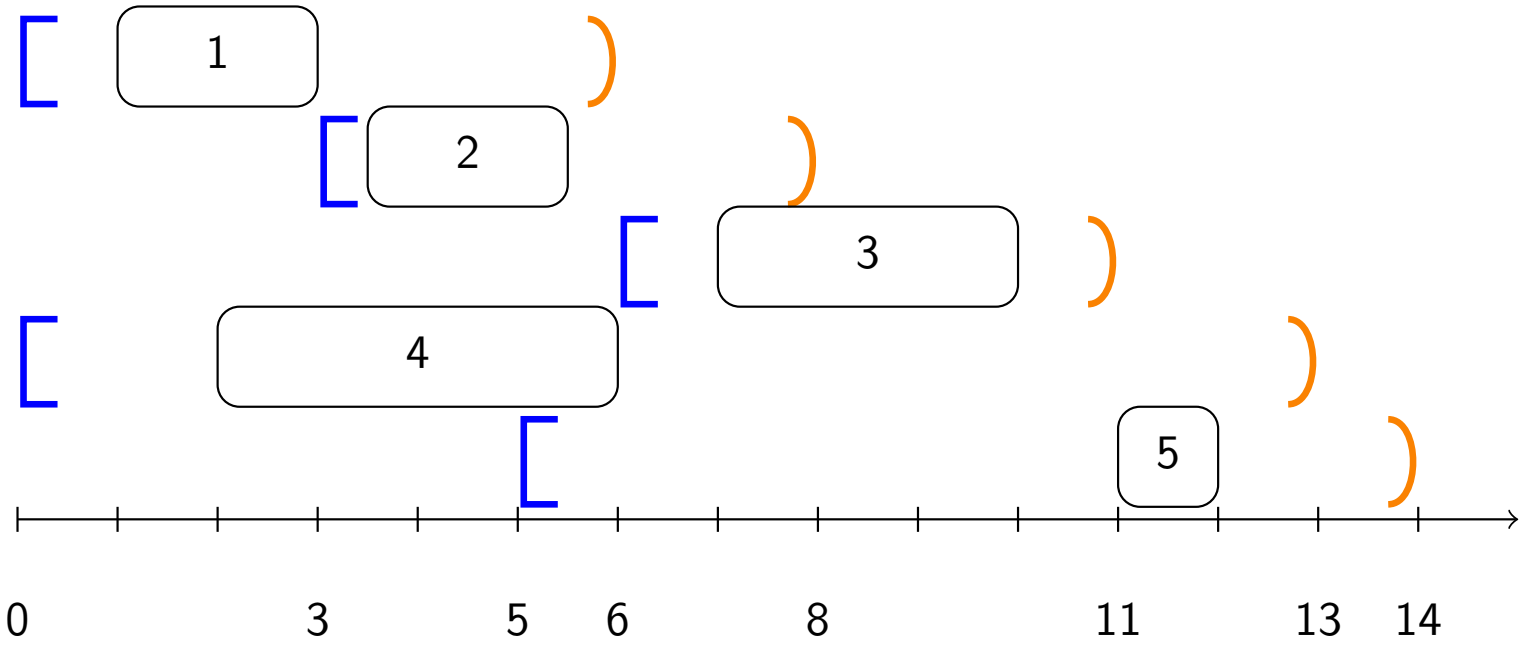
- Key argument 1: A schedule is defined by a path on the state graph.
- Feasibility: finding a path from a source to a sink
- Optimization: finding a minimum cost path from a source to a sink
- Key Argument 2 : Bounding the number of states.

Parameters: A single machine problem example

We consider $1|prec, r_i, \bar{d}_i|C_{max}$ usual parameters:

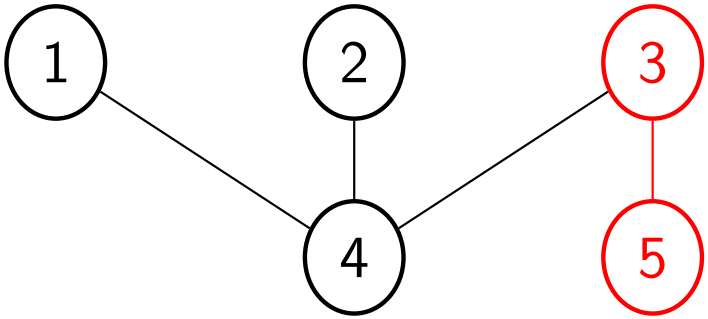
- C_{max}
- Slack $\sigma = \max_i(\bar{d}_i - r_i - p_i)$
- $p_{max} = \max_i p_i$





Compatibility graph

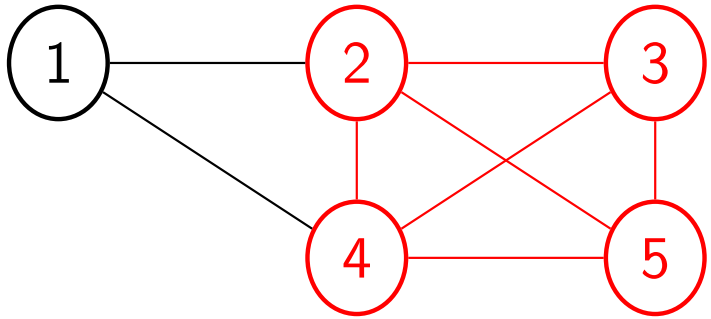
Compatibility graph of precedence constraints



$maxclique = w(G)$ (width of the precedence graph)

$P|prec, p_j = 1|C_{max} \leq D$ is XNLP-complete[Bodlaender et al, 2022]

Compatibility graph of time windows



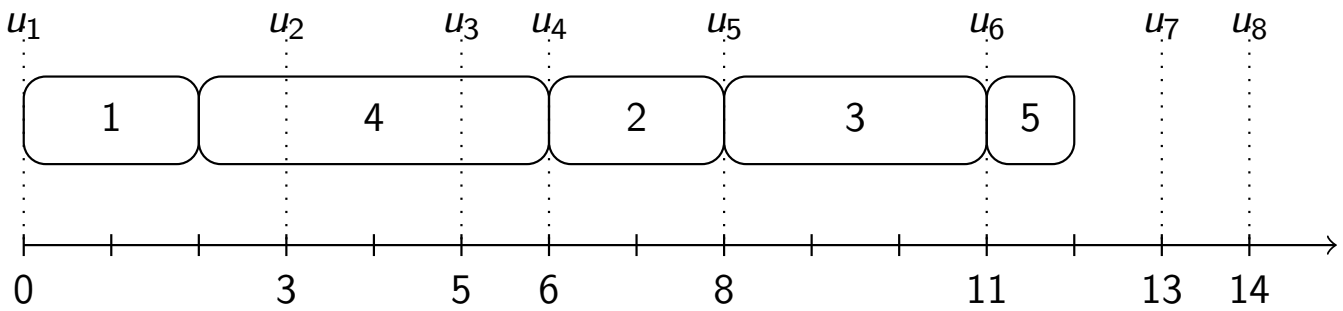
maxclique is the nb or overlaping intervals

$Pathwidth = maxclique - 1 = 3$

$P|prec, p_i = 1, r_i, d_i|\star$ FPT [Munier Kordon 2021]

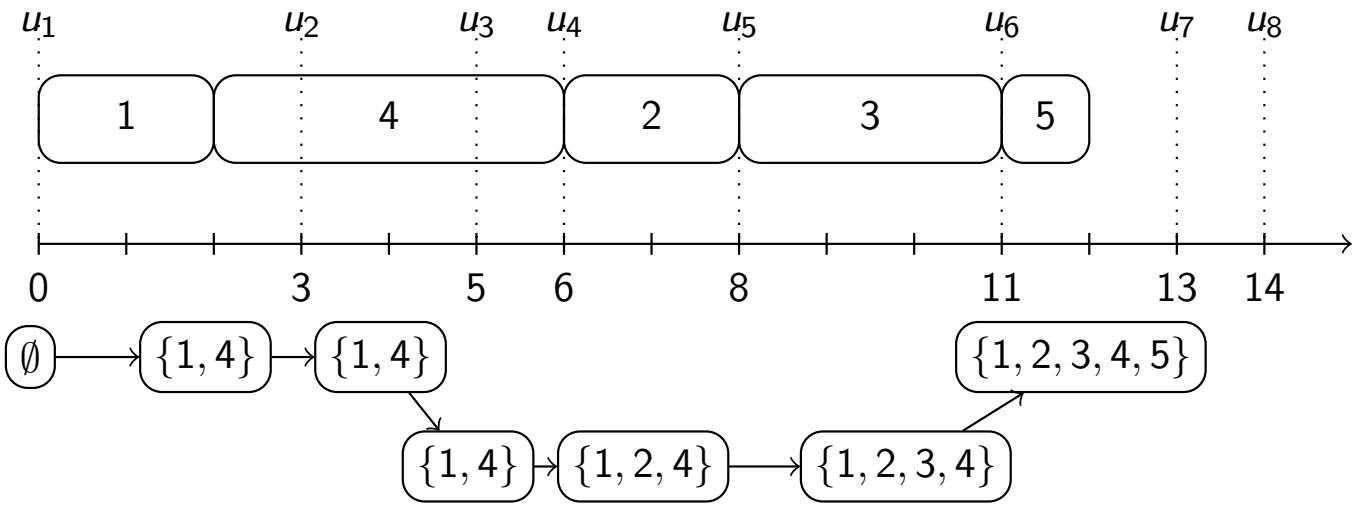
Single machine with pathwidth μ

Sorting the end points of time windows $u_1, \ldots, u_K$



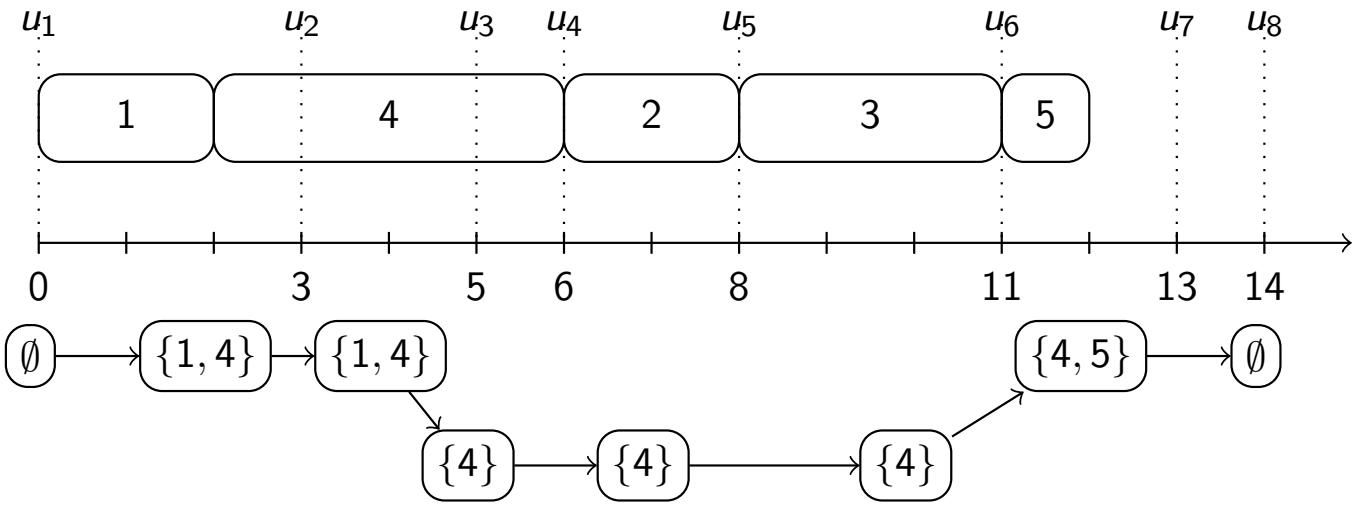
Single machine with pathwidth μ

Recording the set of scheduled jobs before each u_{k+1}

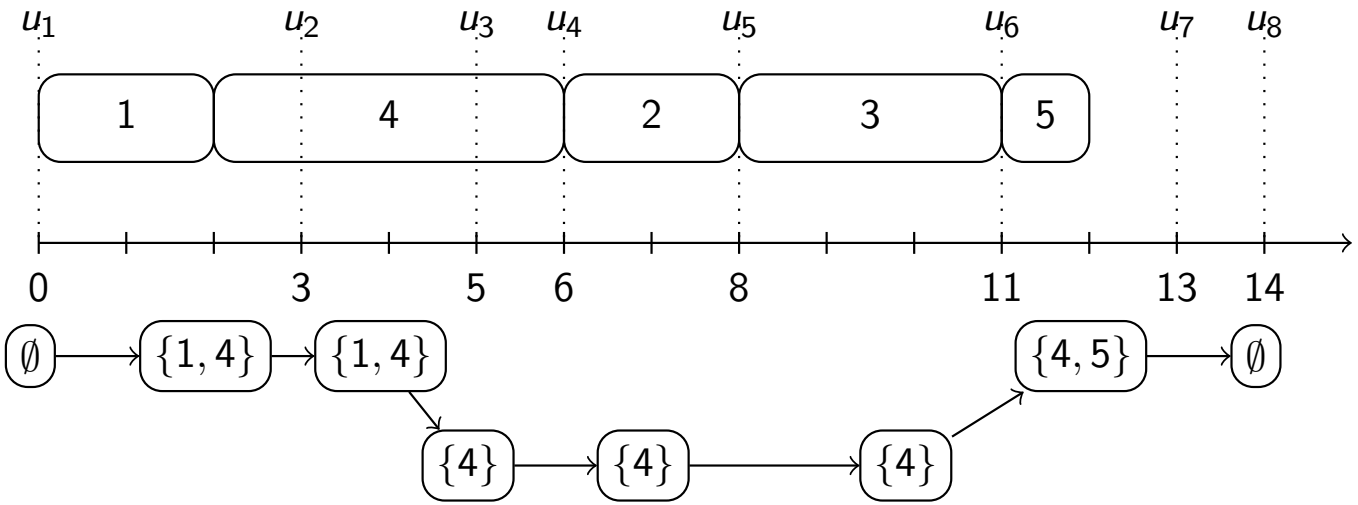


Single machine with pathwidth μ

Removing jobs that must end before u_{k+1}



Single machine with pathwidth μ



State L_k of stage k

L_k Set of jobs started before u_{k+1} and with deadline $> u_{k+1}$ Evaluation of a state: minimum makespan of a feasible schedule of L_k plus jobs with deadlines $\leq u_{k+1}$.

Results

Remark: nb of jobs i with $r_i \leq u_k < u_{k+1} < d_i$ is bounded by $\mu + 1$
In a state L_k of stage k, L_k is a subset of a set of size at most $\mu + 1$.

Theorem

- *The number of states is* $\mathcal{O}(n \cdot 2^{\mu+2})$.
- *The number of arcs is* $\mathcal{O}(n \cdot 2^{2(\mu+2)})$
- *Finding the optimal schedule is FPT with respect to* μ

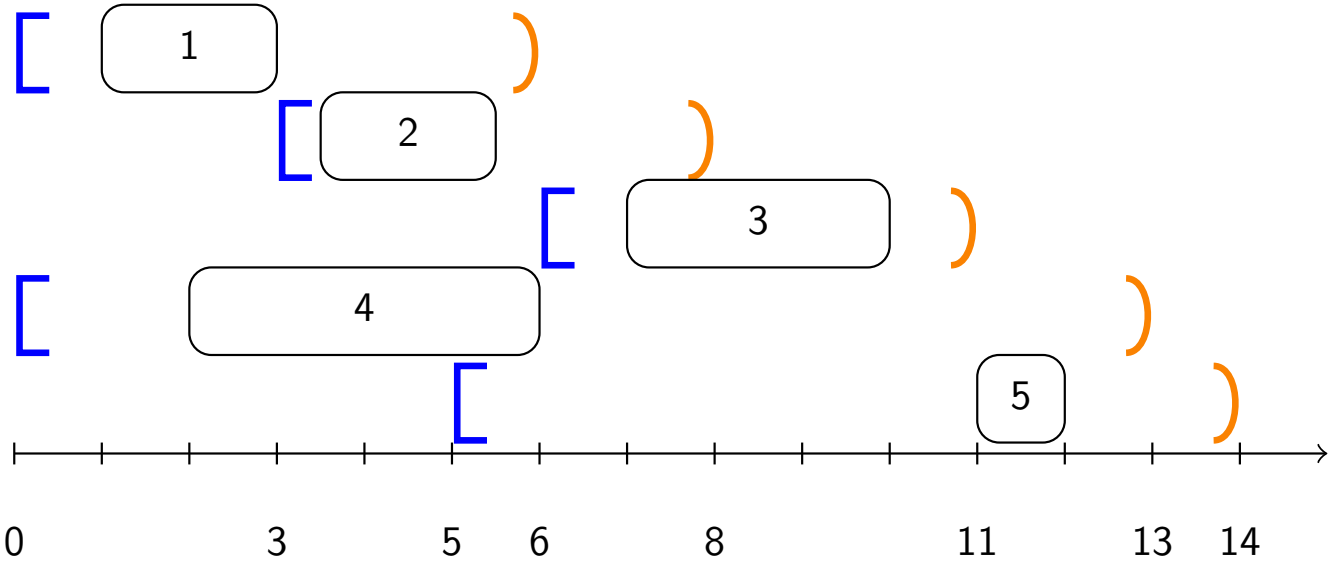
q-proper level parameter

Proper set Π_i of a job i

Set of jobs whose interval strictly encloses $[r_i, \bar{d}_i)$

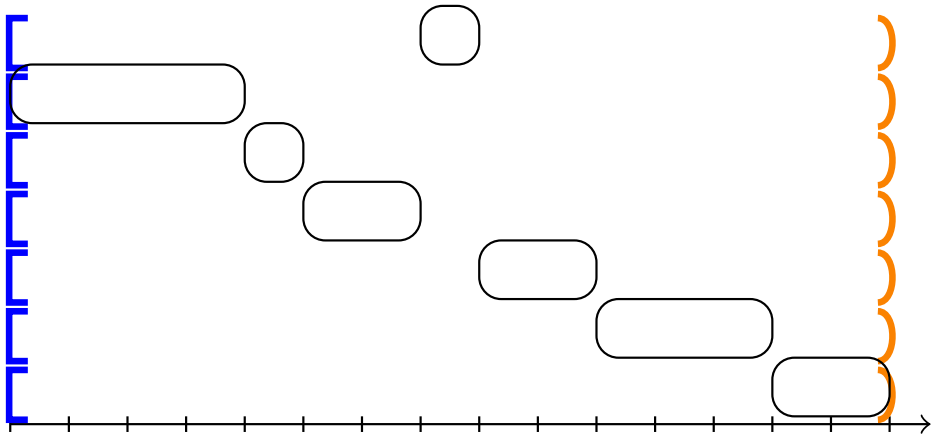
q-proper parameter

$q = \max_i |\Pi_i|$

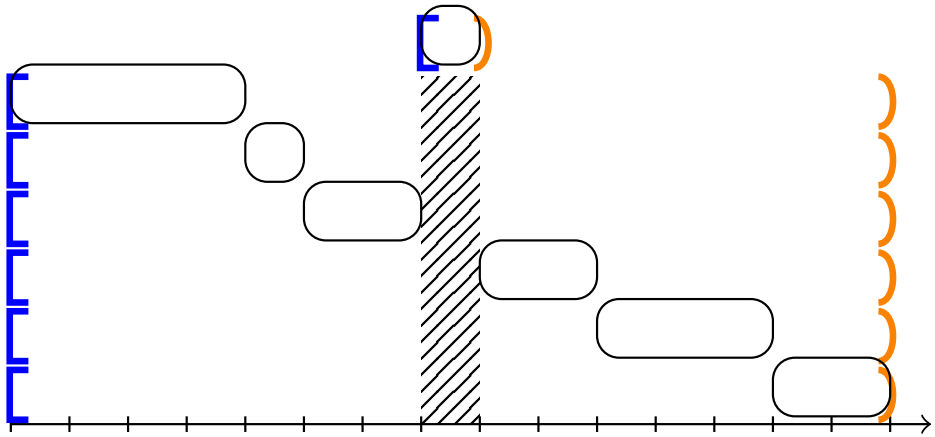


$\Pi_1 = \Pi_2 = \Pi_3 = \emptyset,$
$\Pi_2 = \{4\},$
$\Pi_3 = \{4, 5\},$
$q = 2$

q-proper level motivation



$\mu = n - 1$, $q = 0$, Easy problem



$\mu = n - 1$, $q = n - 1$, NP-hard problem

Assumptions

Consistency of time windows

We assume release times and deadlines consistent with precedence constraints: if $i \to j$ then $r_j \geq r_i + p_i \wedge \bar{d}_i \leq \bar{d}_j - p_j$

Indexation

We assume jobs are indexed such that : if $i < j$ then

$$\bar{d}_i < \bar{d}_j \vee ((\bar{d}_i = \bar{d}_j) \wedge (r_i \leq r_j)$$

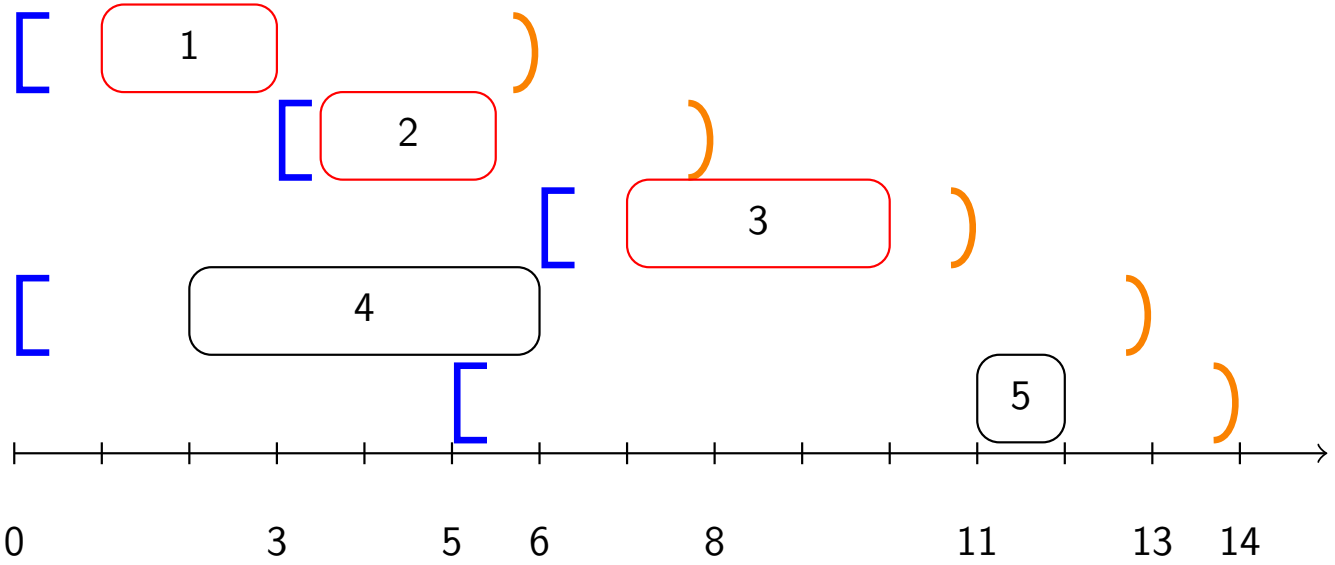
Summits

Definition

A summit is a job i such that the time window $[r_i, \bar{d}_i)$ does not stricly enclose another time window:

$$\forall j \neq i, i \notin \Pi_j$$

Increasing sequence $s_1 < \ldots < s_K$ of summits



Summits

$s_1 = 1$

$s_2 = 2$

$s_3 = 3$

Structure of a schedule : a dominance rule

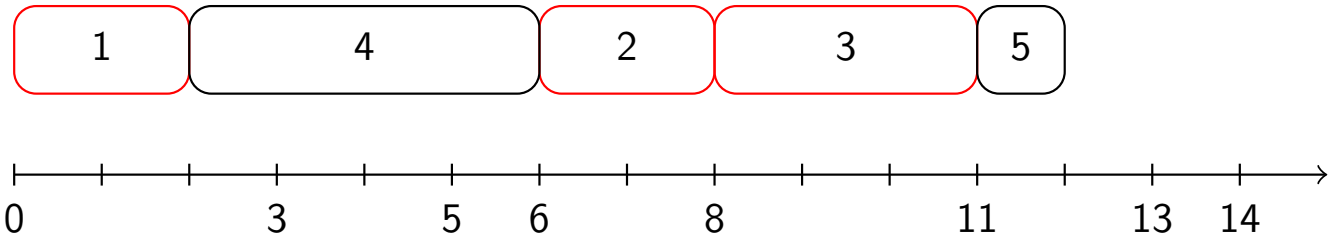
Dominance rule lemma

If there is a feasible schedule then there is a minimum makespan schedule σ of the form $T_1 s_1 \dots T_N s_N T_{N+1}$ where:

a. every job in sequence T_1 is in set Π_{s_1},

b. for all k in $[2, N]$ every job in sequence T_k is in set $\Pi_{s_{k-1}} \cup \Pi_{s_k}$, and

c. every job in sequence T_{N+1} is in set Π_{s_N}.



$T_1 = \emptyset, T_2 = \{4\}, T_3 = \emptyset, T_4 = \{5\}$

Dominance rule detailed

Lemma

In a dominant schedule $T_1 s_1 \dots T_N s_N T_{N+1}$,

- *All jobs with index* $< s_k$ *are performed before* s_k
- $T_k = C_{k-1} A_k B_k$

Where

- $C_{k-1} = T_k - \Pi_{s_k}$ *scheduled by increasing index*
- $A_k = T_k \cap \Pi_{s_{k-1}} \cap \Pi_{s_k}$ *scheduled by increasing index*
- $B_k = T_k - \Pi_{s_{k-1}}$ *scheduled by increasing* r_i

Inspired by a similar decomposition for problems without prec:

J. Erschler, Gérard Fontan, Colette Mercé, and François Roubellat. *A new dominance concept in scheduling n jobs on a single machine with ready times and due dates.* Oper. Res., 31(1):114–127, 1 (1983)

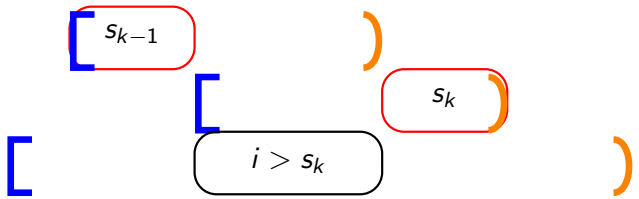
Dynamic programming

- A state of stage k records the minimum information about jobs performed before summit s_k
- Only jobs with index $> s_k$ are useful to store

Lemma

$$L_k = \bigcup_{1 \leq h \leq k} T_h \cap \{s_k + 1 \ldots, n\}$$

$$L_k \subset \Lambda_k = \bigcup_{1 \leq h \leq k} \Pi_{s_h} \cap \{s_k + 1 \ldots, n\} = \Pi_{s_k}$$



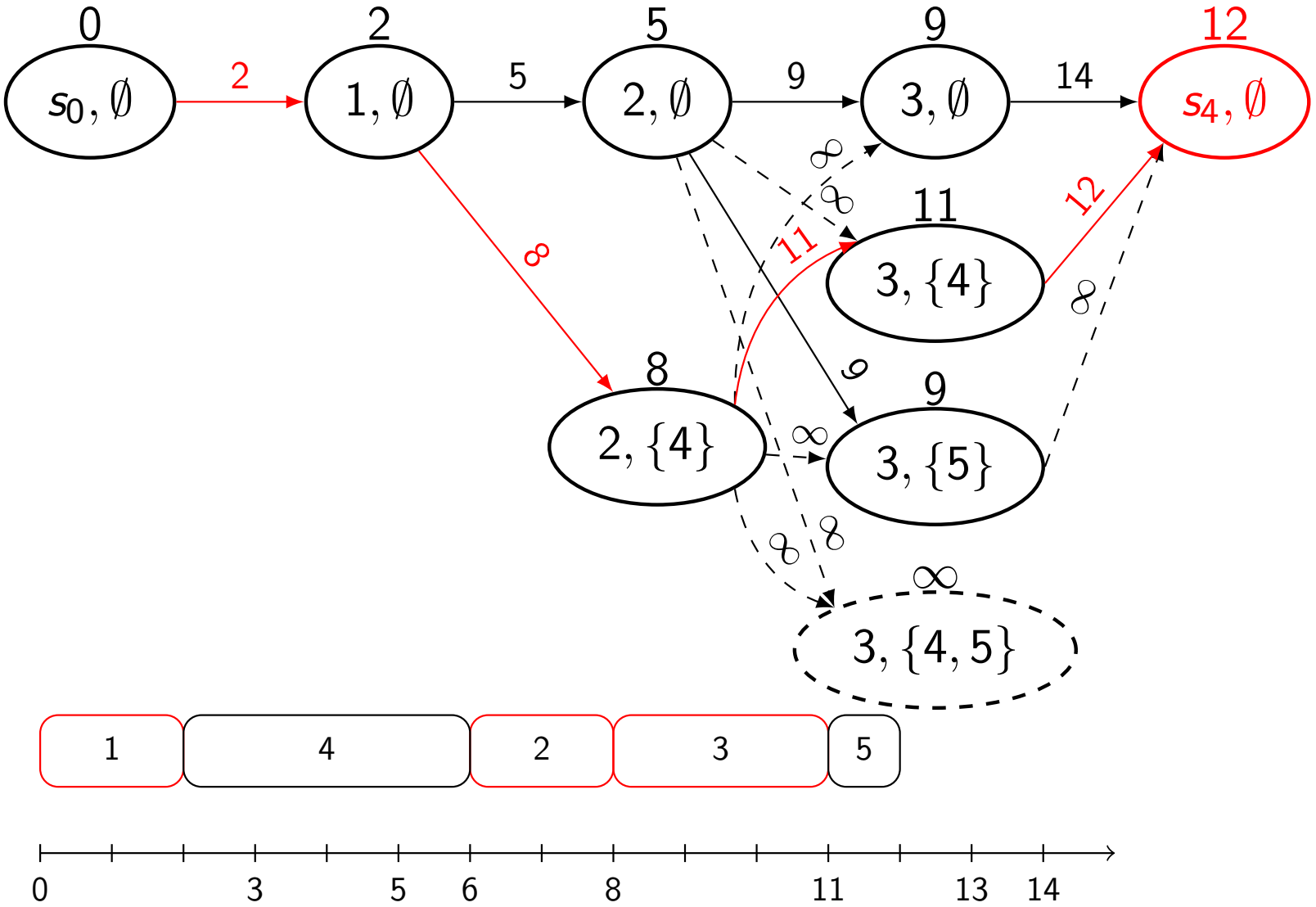
Results

- Cardinality of the set Λ_k bounded by q
- Evaluation of a state L_k of stage k : minimum makespan of a schedule of $\{1, \ldots, s_k\} \cup L_k$ ending by s_k.
- This evaluation can be performed knowing the evaluation of states of the lower stages.

Theorem

- *The number of states is* $\mathcal{O}(n2^q)$
- *Solving* $1|prec, r_i, \bar{d}_i|C_{max}|$ *is FPT with respect to* q.

State graph


0
$s_0, \emptyset$
2
2
$1, \emptyset$
5
5
$2, \emptyset$
9
9
$3, \emptyset$
14
12
$s_4, \emptyset$
8
∞
∞
11
$3, \{4\}$
12
11
∞
8
$2, \{4\}$
∞
9
$3, \{5\}$
∞
∞
∞
$3, \{4, 5\}$
1
4
2
3
5
0
3
5
6
8
11
13
14

Relations between parameters

Lemma

In any instance of $1|prec, r_i, \bar{d}_i|*, \quad q \leq \mu.$

Corollary

A FPT algorithm for q there is a FPT algorithm for parameter μ. If a problem is para-NP-hard for parameter μ, it will be also for parameter q

Lemma

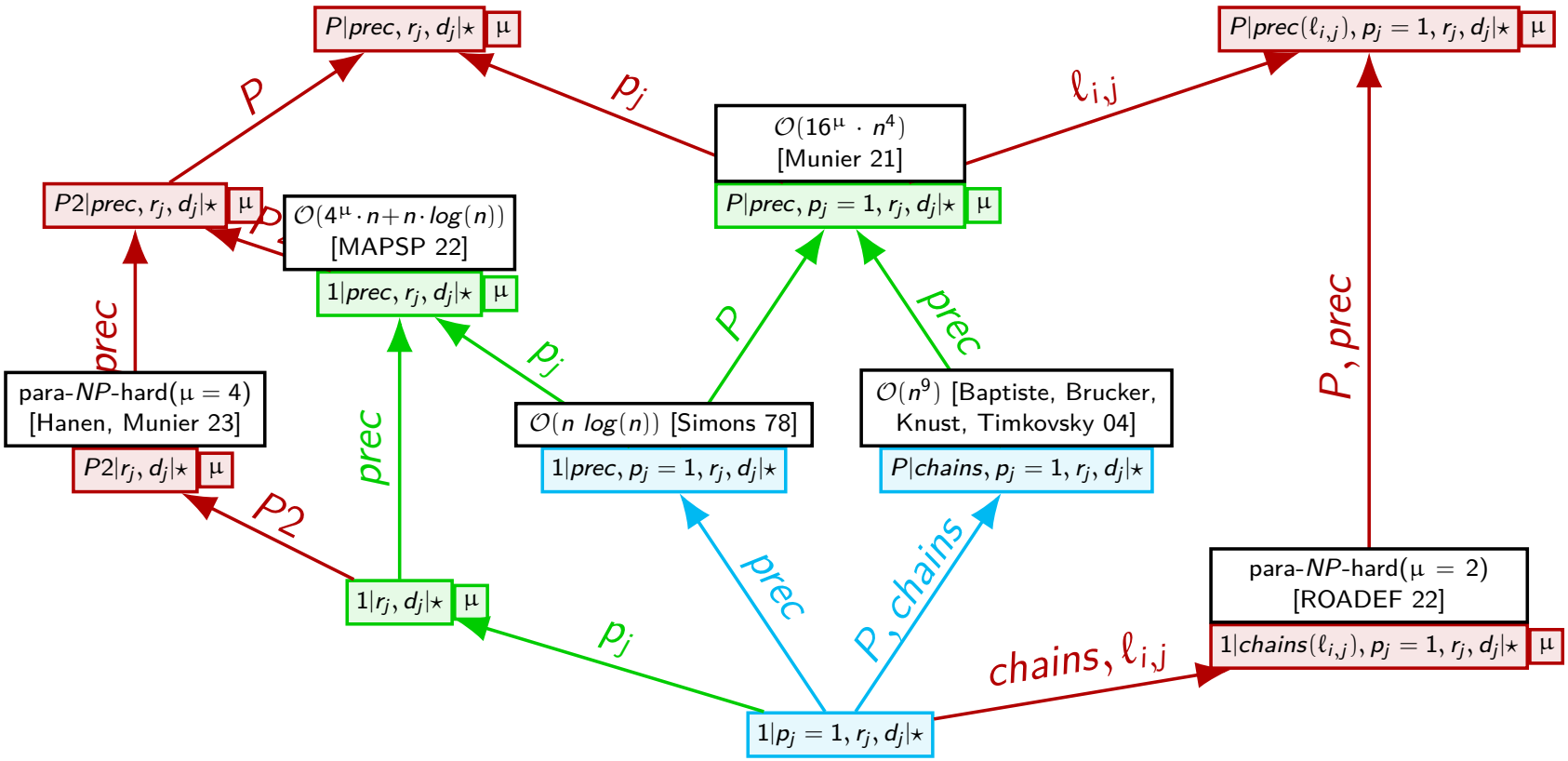
In any feasible instance of $1|prec, r_i, \bar{d}_i|* \ \mu \leq 2\sigma$ *(slack);*

Corollary

A FPT algorithm for μ is a FPT algorithm for parameter σ.

Map

Parameter μ



Concluding remarks

- We provided two FPT algorithms for $1|prec, r_i, \bar{d}_i|C_{max}$ with parameters μ and q.
- Currently studying $P|tree, p_i = 1, r_i, \bar{d}_i|*$ with respect to q
- Building maps for scheduling relevant parameters

Challenging questions for the study of parameterized complexity of scheduling problems :

- Are FPT algorithms efficient in practice?
- What about Kernelization techniques?

Maher Mallem, Claire Hanen, Alix Munier-Kordon:
Parameterized Complexity of Single-machine Scheduling with Precedence, Release Dates and Deadlines, MAPSP2022
A new structural parameter on single machine scheduling with release dates and deadlines, ISCO 2024