Scheduling Monotone Moldable Jobs in Linear Time

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Moldable Job Scheduling

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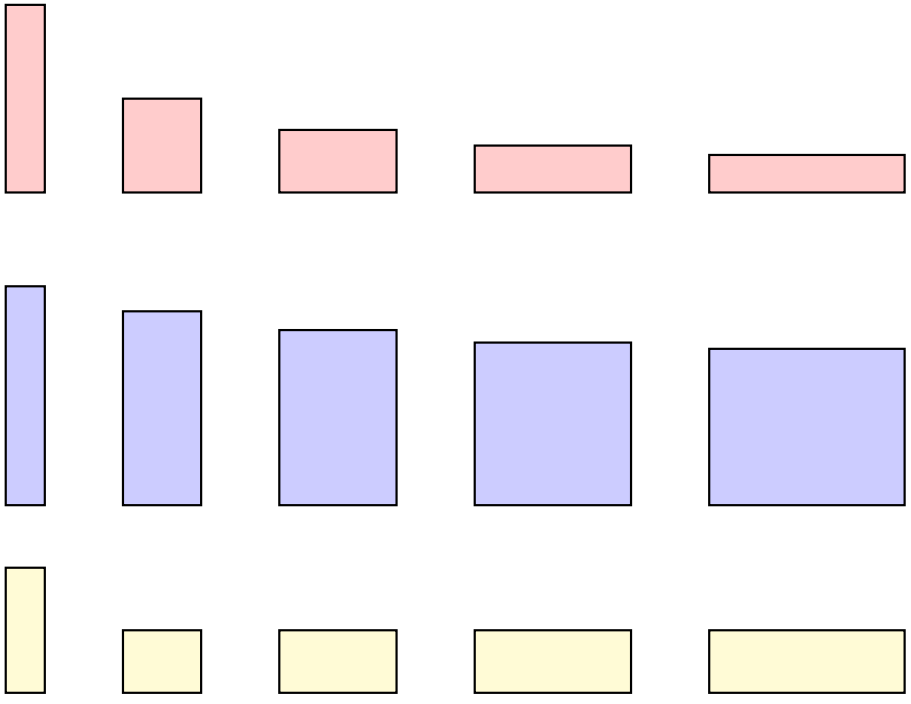
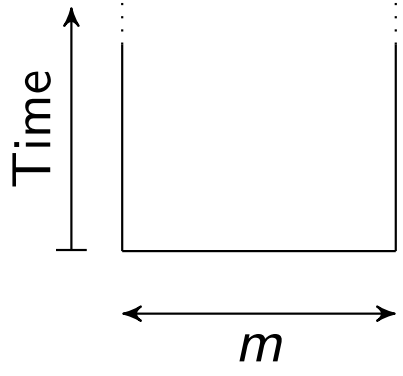
- m machines
- job j has processing time $t_j(k)$ when processed on k machines
  work $w_j(k) = k \times t_j(k)$





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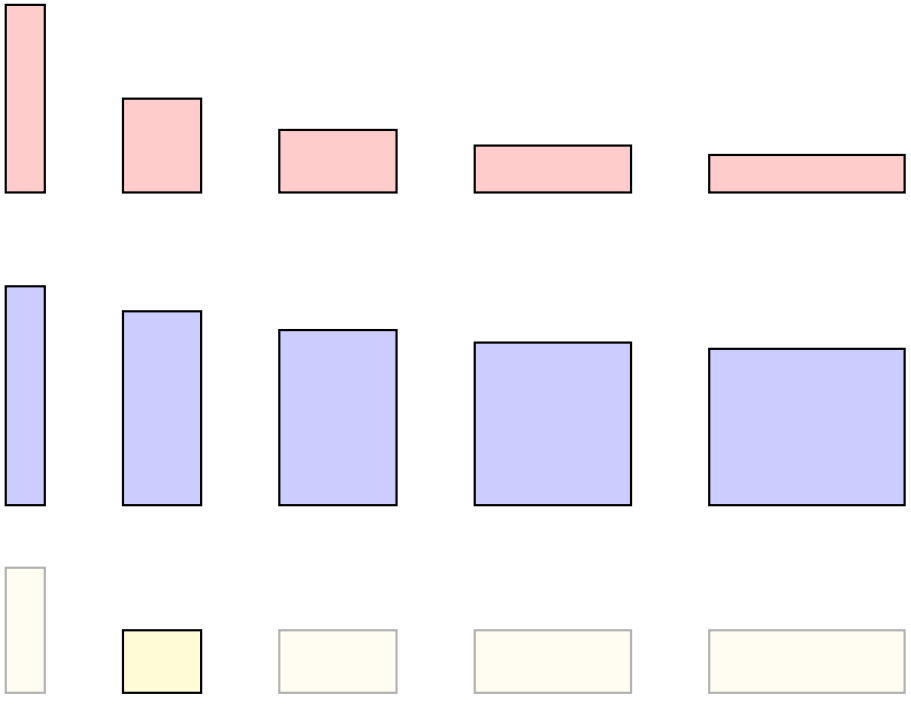

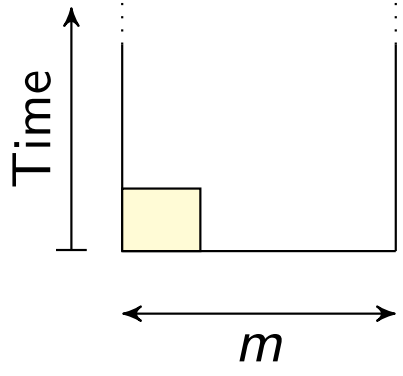

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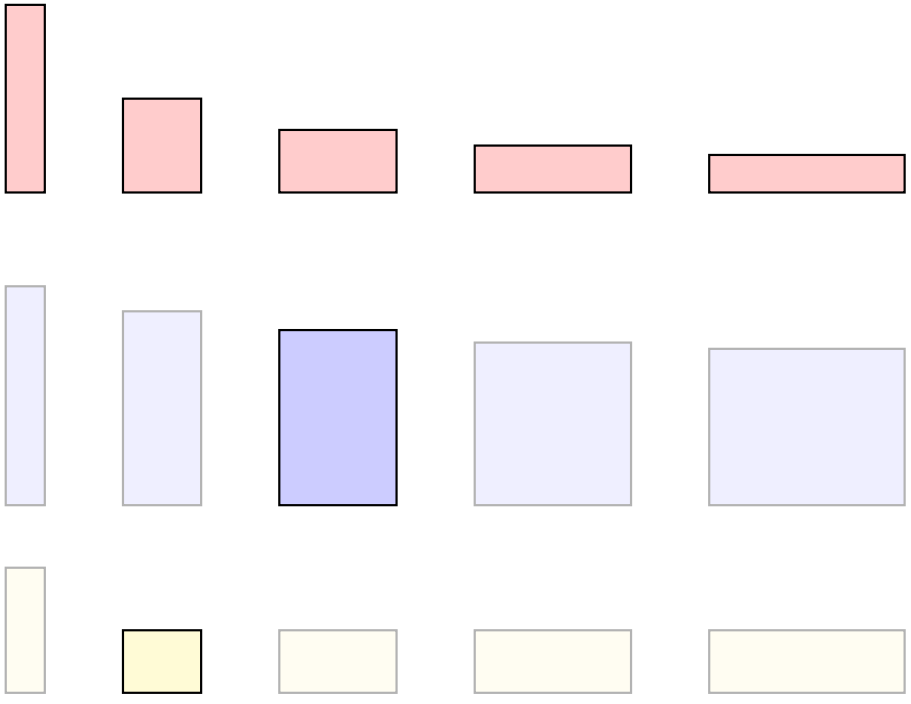

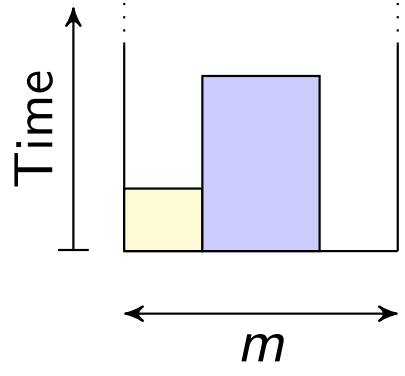

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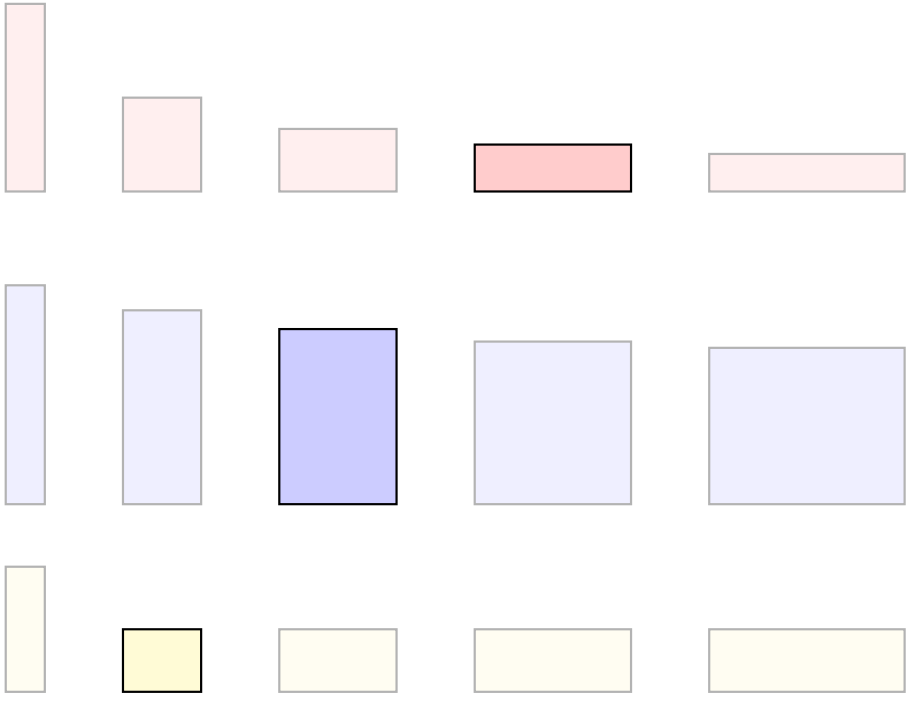

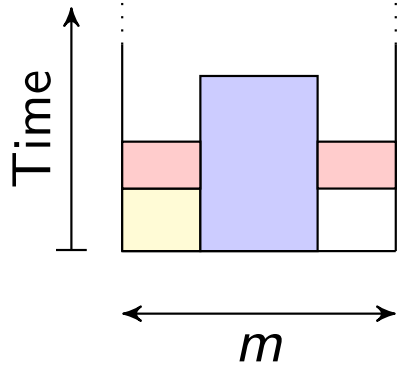

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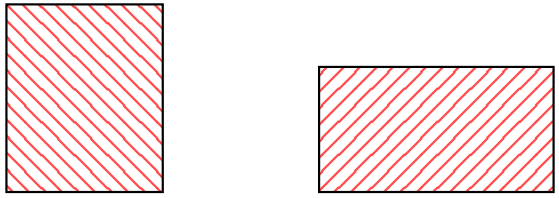
Monotone Jobs

Definition

A job j is monotone iff its work function $w_j(k) = k \times t_j(k)$ is non-decreasing.



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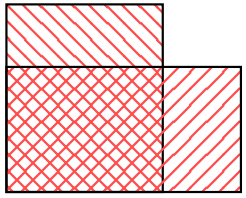

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Encoding the Processing Times

Usual assumption: encode the nm processing times in the input

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- Algorithms can have running time $\mathrm{poly}(n, m)$

Often: compact encoding possible

- m can be exponential in $\|I\|$
- Algorithms should have running time $\mathrm{poly}(n, \log m)$

Known Results & This Work

General Jobs

	m unbounded	$m \leq \|I\|$	m constant
Hardness	$\frac{3}{2} - \varepsilon$ inapprox.	NP-hard for $m \geq 4$	
Approximation	$\frac{3}{2} + \varepsilon$	PTAS	EPTAS

Monotone Jobs

	m unbounded	$m \leq \|I\|$	m constant
Hardness	weakly NP-hard for $m \geq 2$		
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General Jobs

	m unbounded	$m \leq \|I\|$	m constant
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Monotone Jobs

	m unbounded	$m \leq \|I\|$	m constant
Hardness	strongly NP-hard		weakly NP-hard
Approximation	PTAS, fast $\frac{3}{2} + \varepsilon$	PTAS	EPTAS

Compression

Lemma

Let $\rho \in (0, 1/4]$ *and* $k \in [m]$, $k \geq \frac{1}{\rho}$.
Then $\mathrm{t}_j(\lfloor k(1-\rho) \rfloor) \leq (1+4\rho)\,\mathrm{t}_j(k)$.

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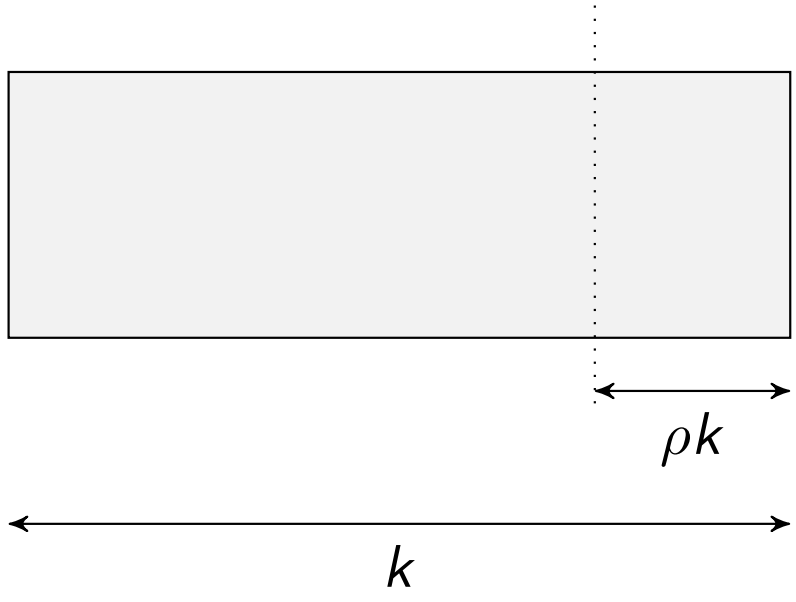

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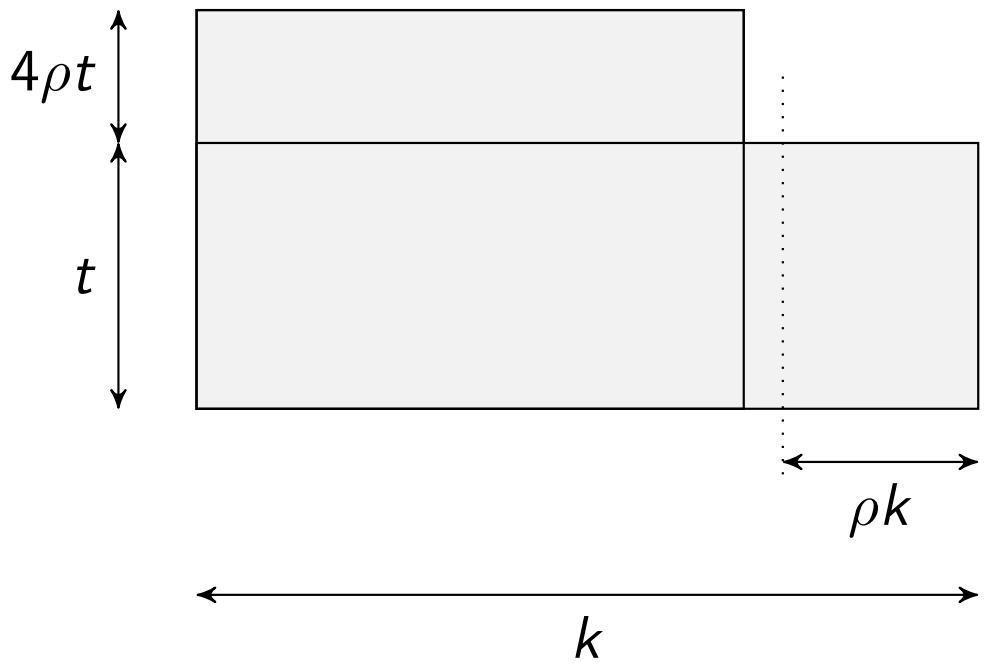

Dual Approximation [Hochbaum, Shmoys, 1987]

- Known bounds $LB \leq \mathrm{OPT} \leq UB$, $\frac{UB}{LB} = \mathrm{O}(1)$
- A c-dual approximation algorithm
  - gets a guess d of the optimum makespan
  - computes a schedule with makespan $\leq cd$
  - if $d < \mathrm{OPT}$: may reject
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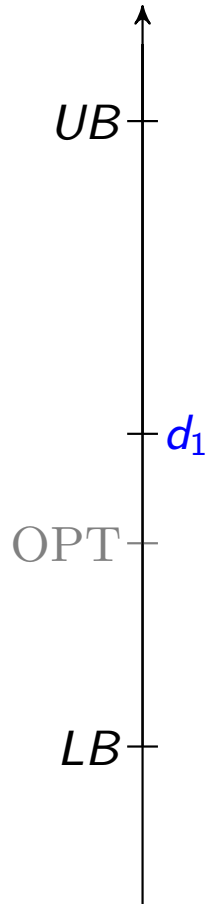

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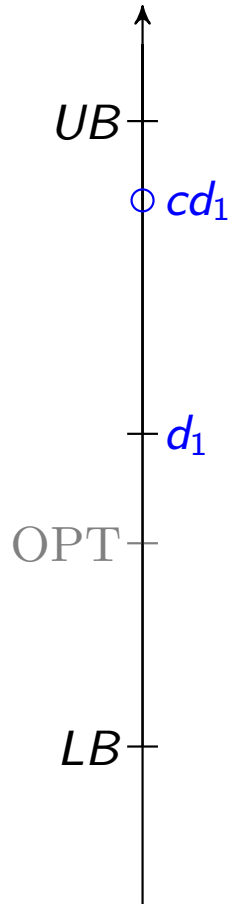

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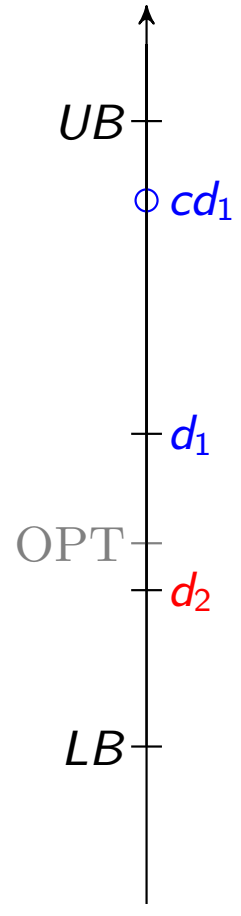

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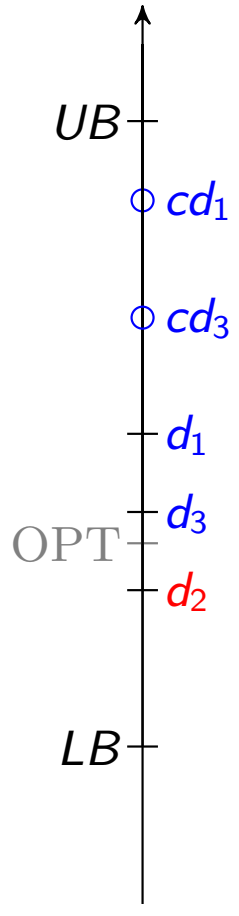

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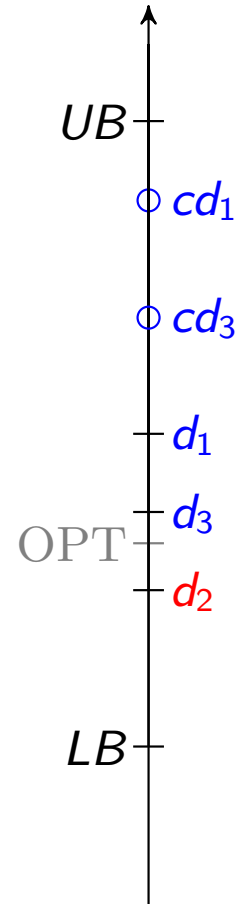
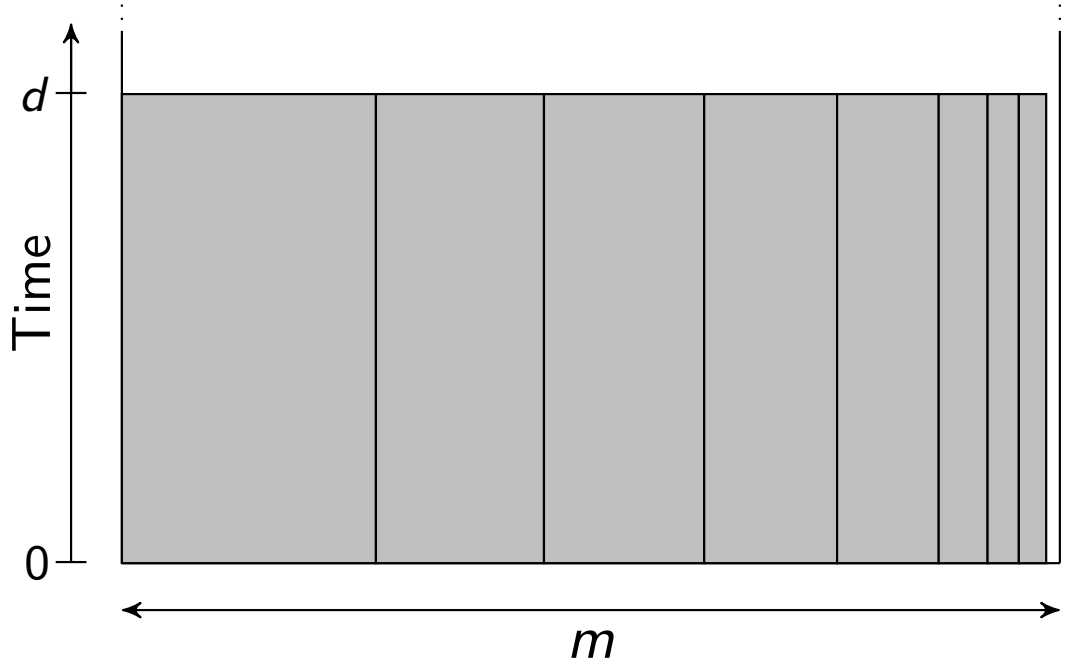
- Use 2-approximate algorithm [Ludwig, Tiwari, 1994] to obtain bounds
  Running time $\mathrm{O}(n \log^2 m)$



FPTAS: $(1+\varepsilon)$-dual algorithm for large m



- Ideal: Fractional schedule with makespan d

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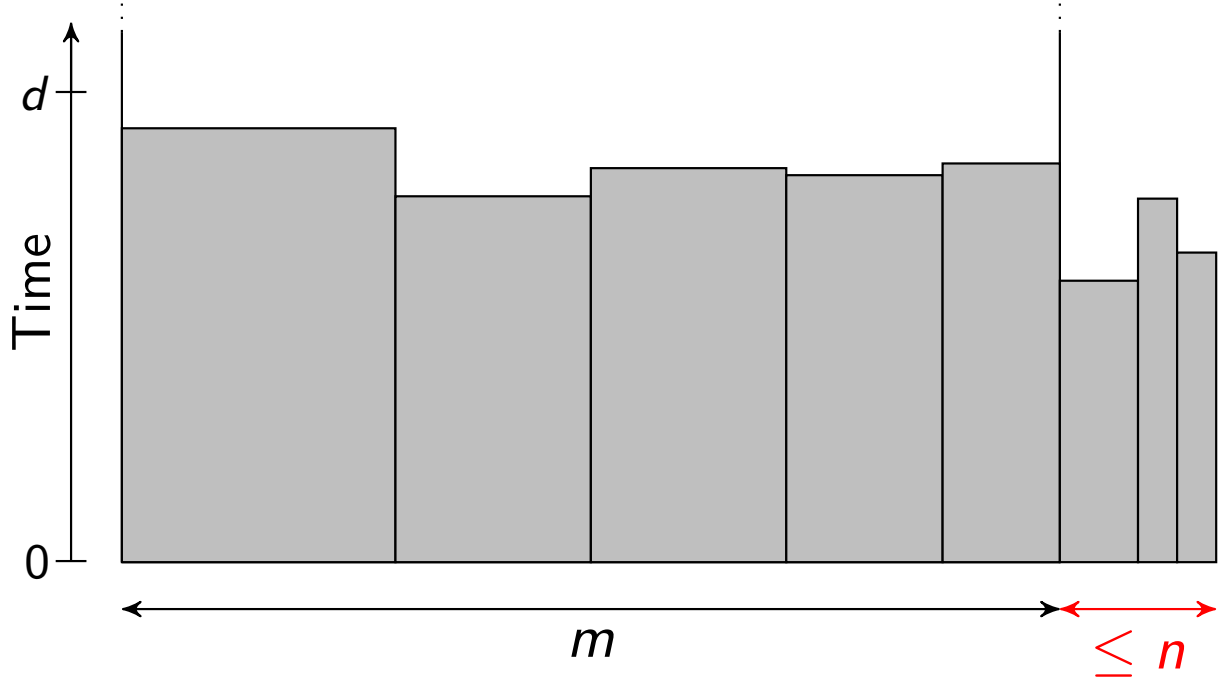

- Schedule all jobs in parallel such that they process just in d time units

$$\gamma(j,d) = \min\{k \in [m] \mid \mathrm{t}_j(k) \leq d\}$$

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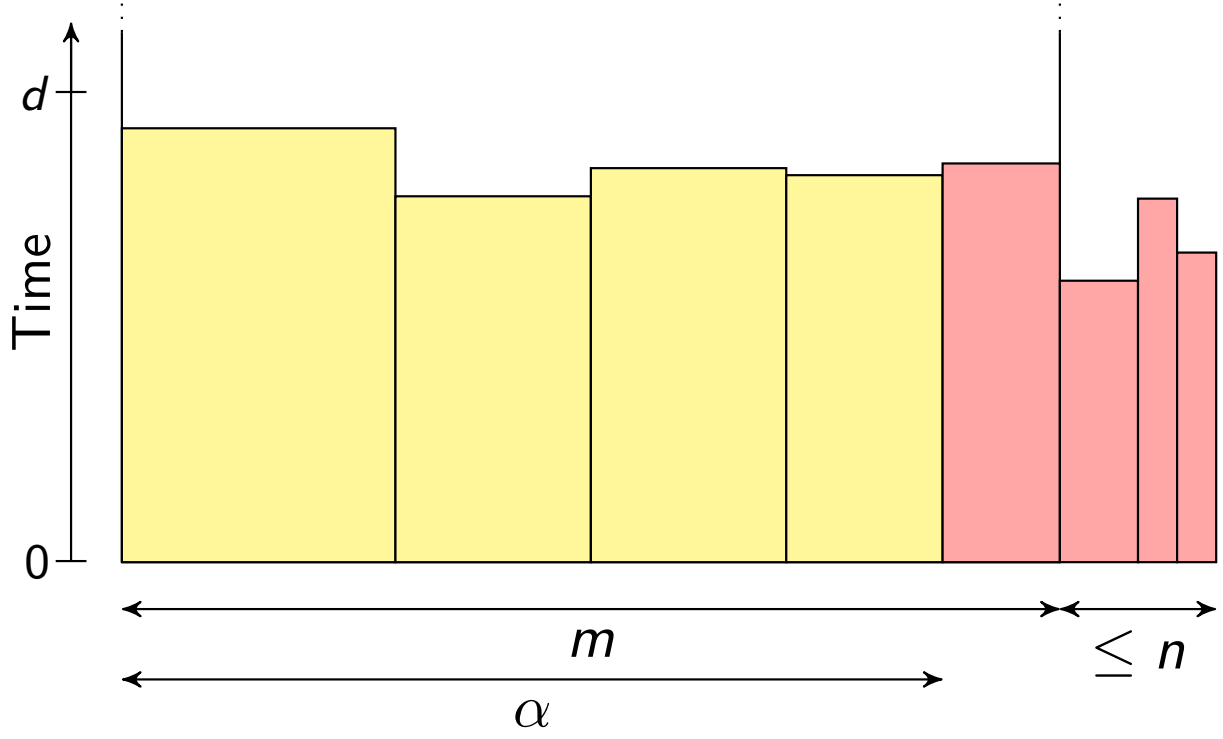

- Schedule all jobs in parallel such that they process just in d time units
- Compress all compressible jobs with factor $\rho = \frac{\varepsilon}{4}$

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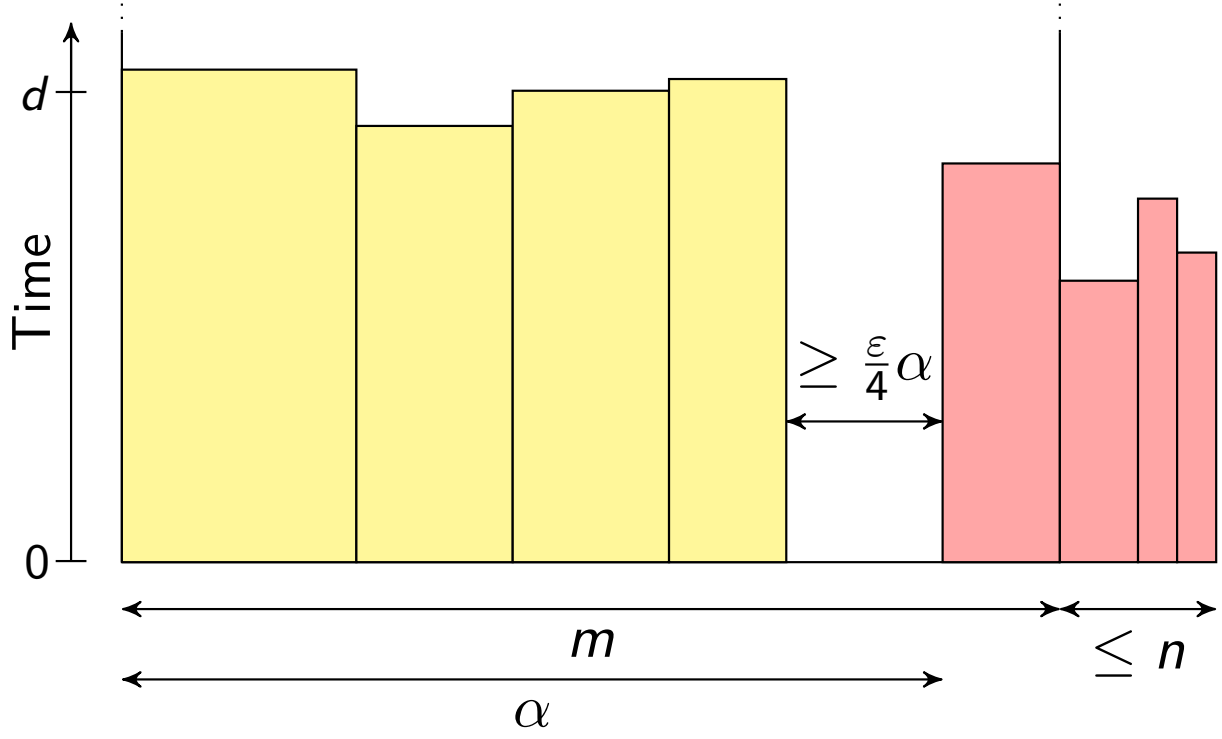

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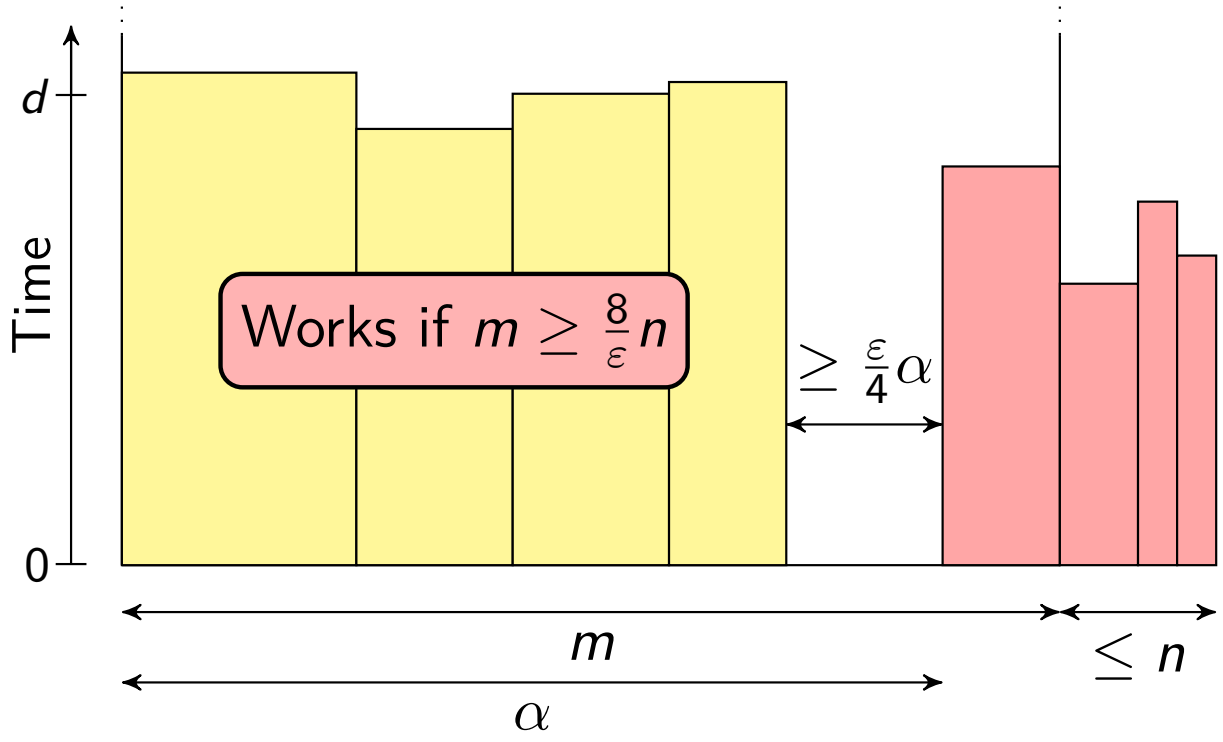

- Schedule all jobs in parallel such that they process just in d time units
- Number of machines used by non-compressible jobs is at most $n/\rho = 4n/\varepsilon$

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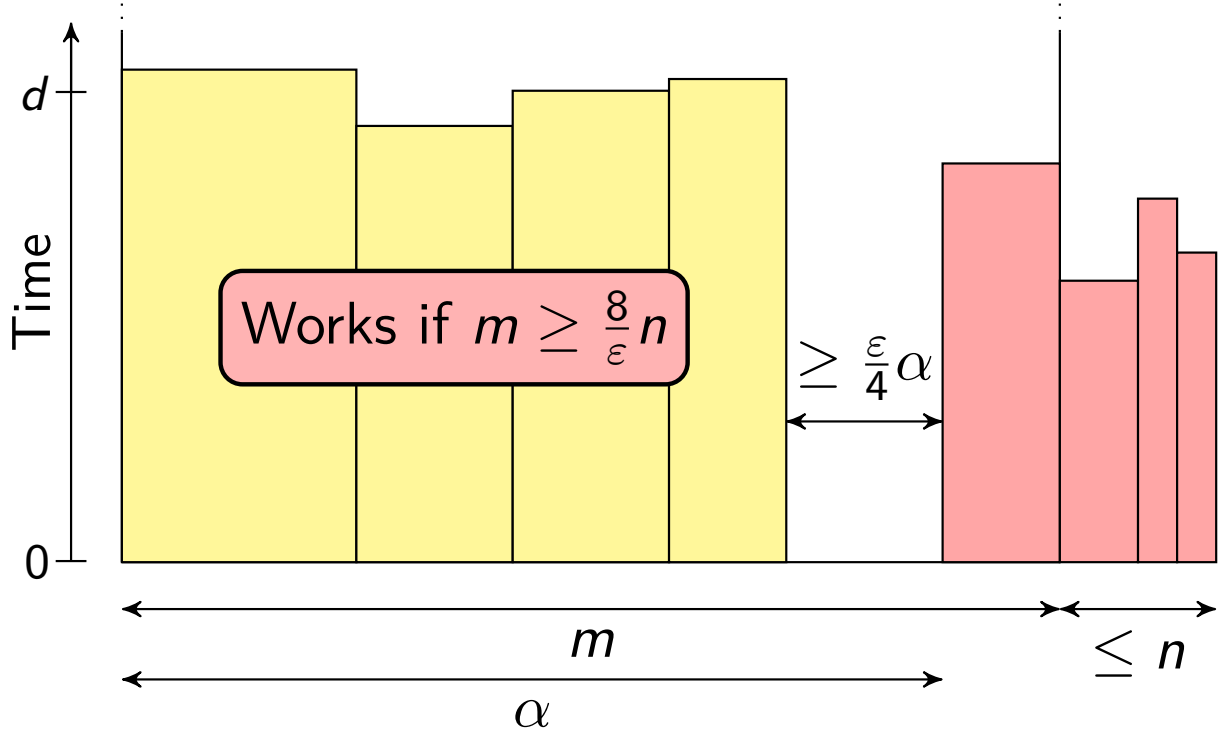

- Schedule all jobs in parallel such that they process just in ~~d~~ $(1+\varepsilon)d$ time units
- Running time $\mathrm{O}(n \log^2(\frac{1}{\varepsilon} + \frac{\log(\varepsilon m)}{\varepsilon}))$

A PTAS: general case

- If $m \geq \frac{8}{\varepsilon} n$, use our $(1+\varepsilon)$-dual algorithm
- Otherwise, $m = \mathrm{O}(\frac{n}{\varepsilon})$; use PTAS for that case [Jansen, Thöle, 2010]

A Linear $(\frac{3}{2} + \varepsilon)$-approximate Algorithm

Improves upon $(\frac{3}{2} + \varepsilon)$-dual approximate algorithm with running time $\mathrm{O}(nm \log(1/\varepsilon))$ [Mounié, Rapine, Trystram, 2007]

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Steps of the Algorithms

1. Remove Small Jobs
2. Find two shelf schedule s.t. total work (including removed jobs) at most md
3. Repair two shelf schedule
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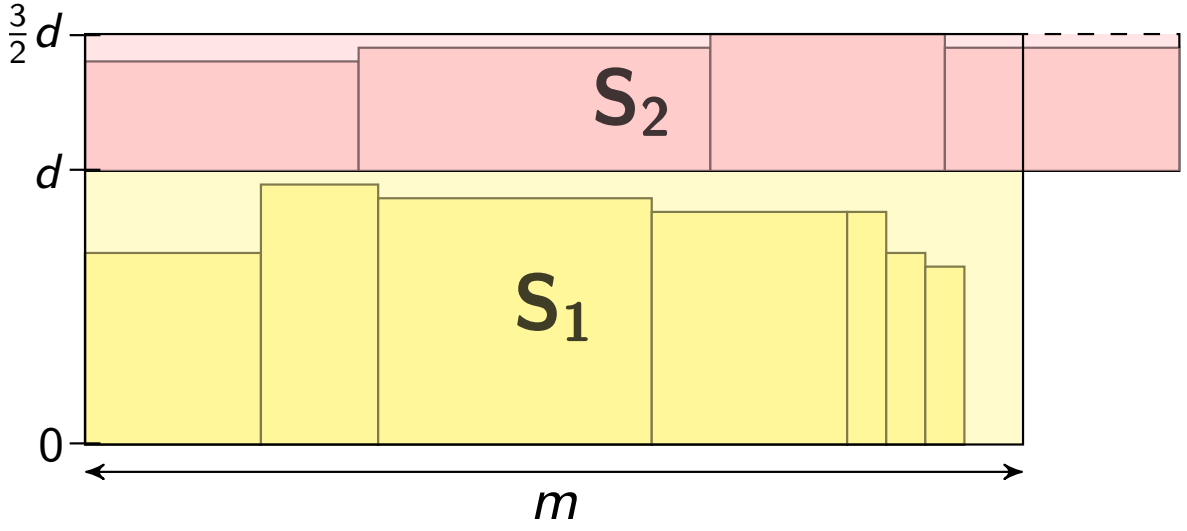
Two Shelf Schedules



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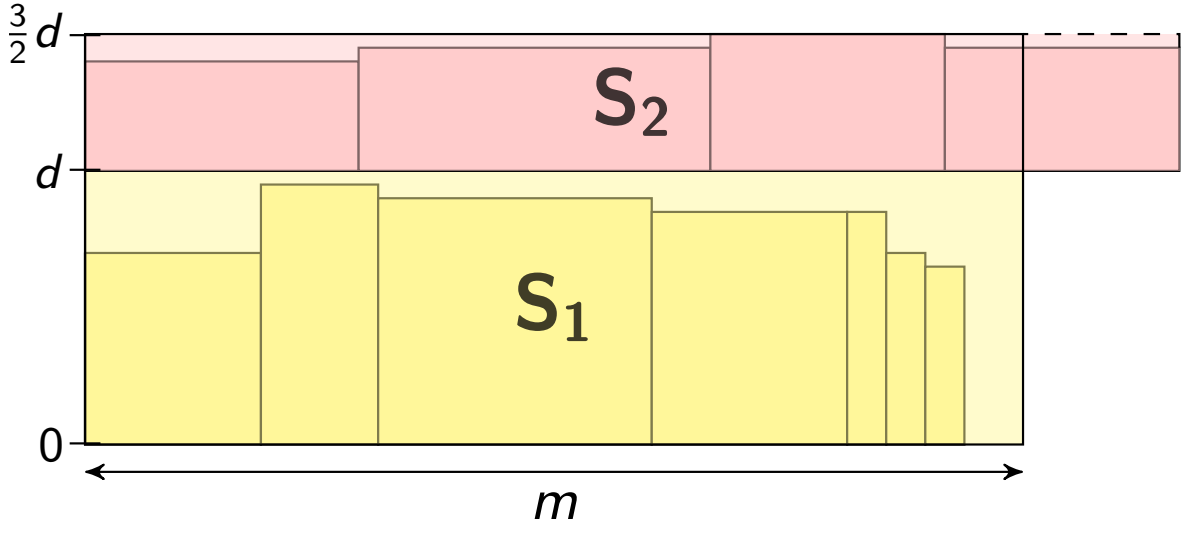

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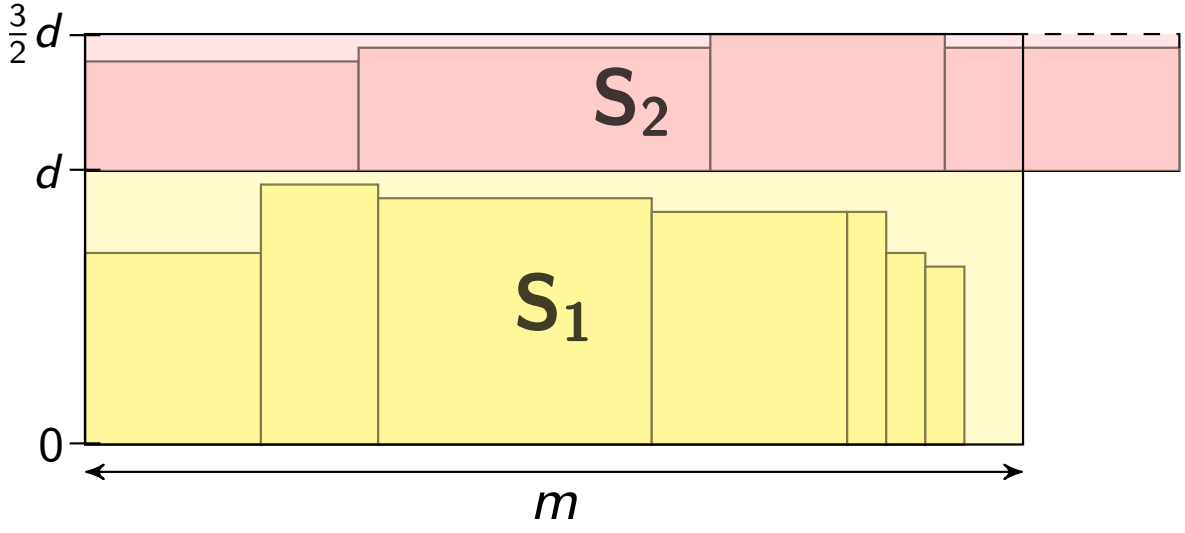

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$$\max_{J' \subseteq J_B} \sum_{j \in J'} w_j \left(\gamma\left(j, \frac{d}{2}\right) \right) - w_j(\gamma(j,d))$$

$$\sum_{j \in J'} \gamma(j,d) \leq m$$

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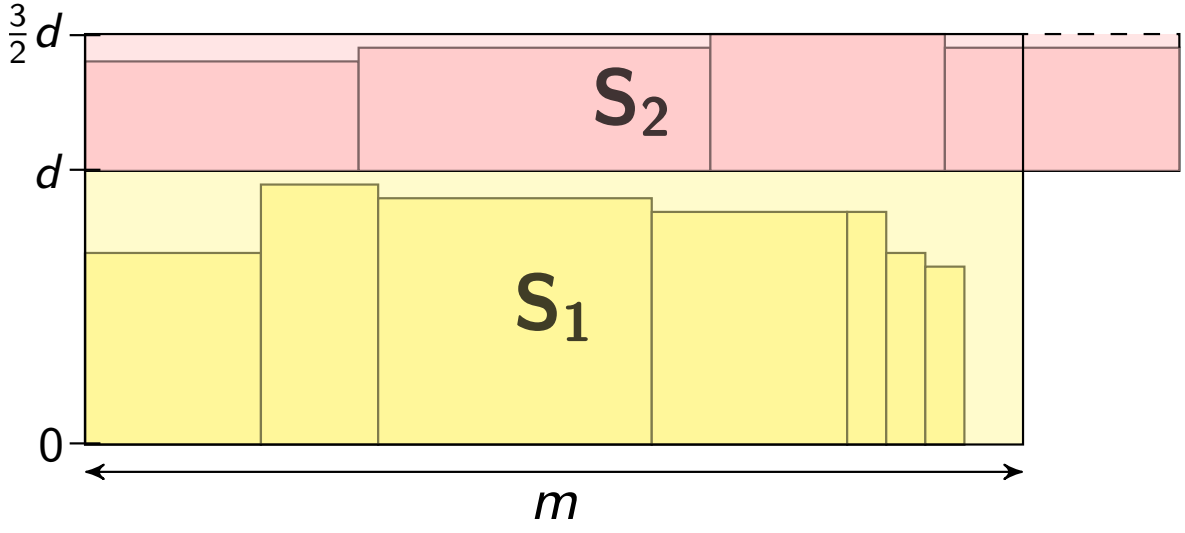

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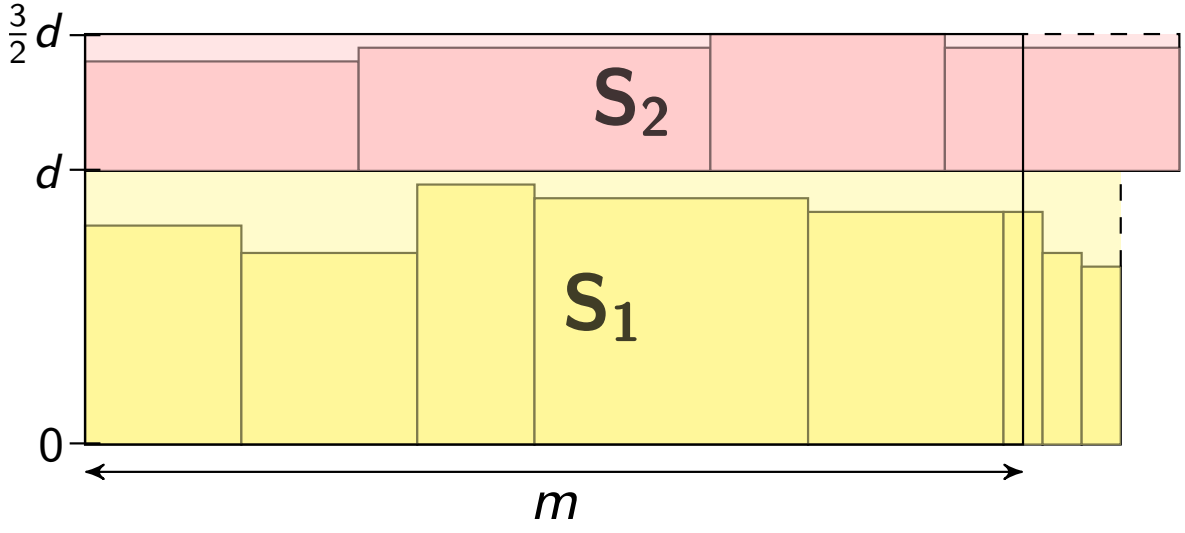

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$$\sum_{j \in J' \cap J^c} (1-\rho)\,\gamma(j,d) + \sum_{j \in J' \setminus J^c} \gamma(j,d) \leq m$$

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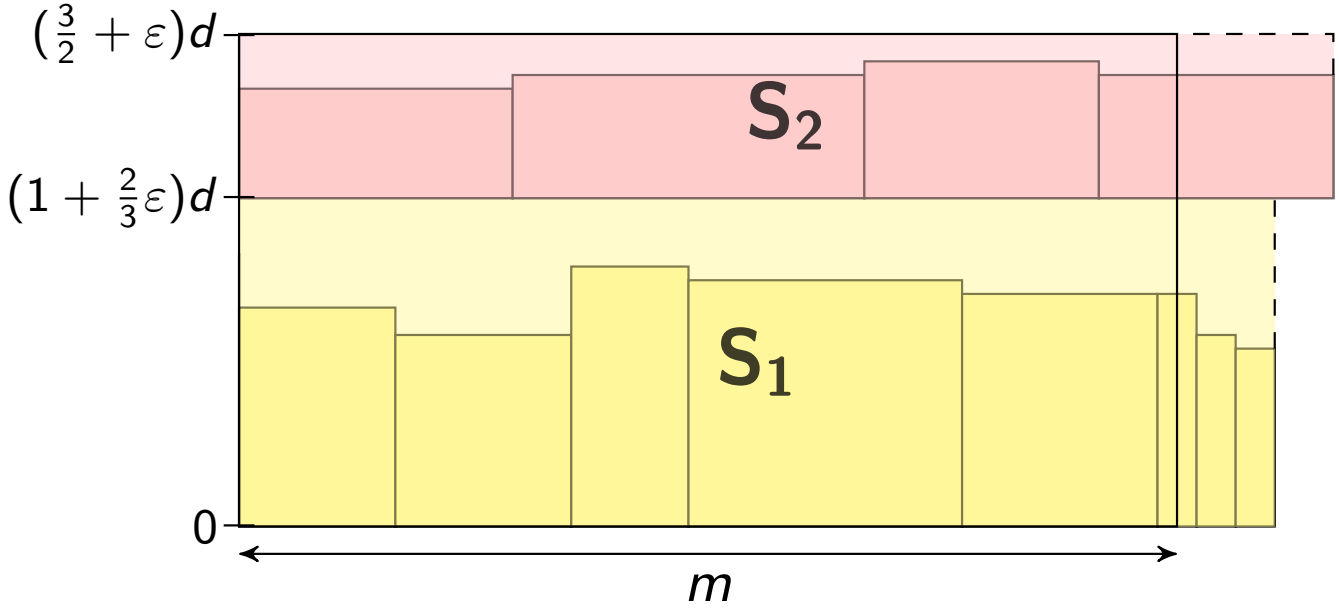

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- Increase shelf height – compressed jobs fit, allows for work slightly above md

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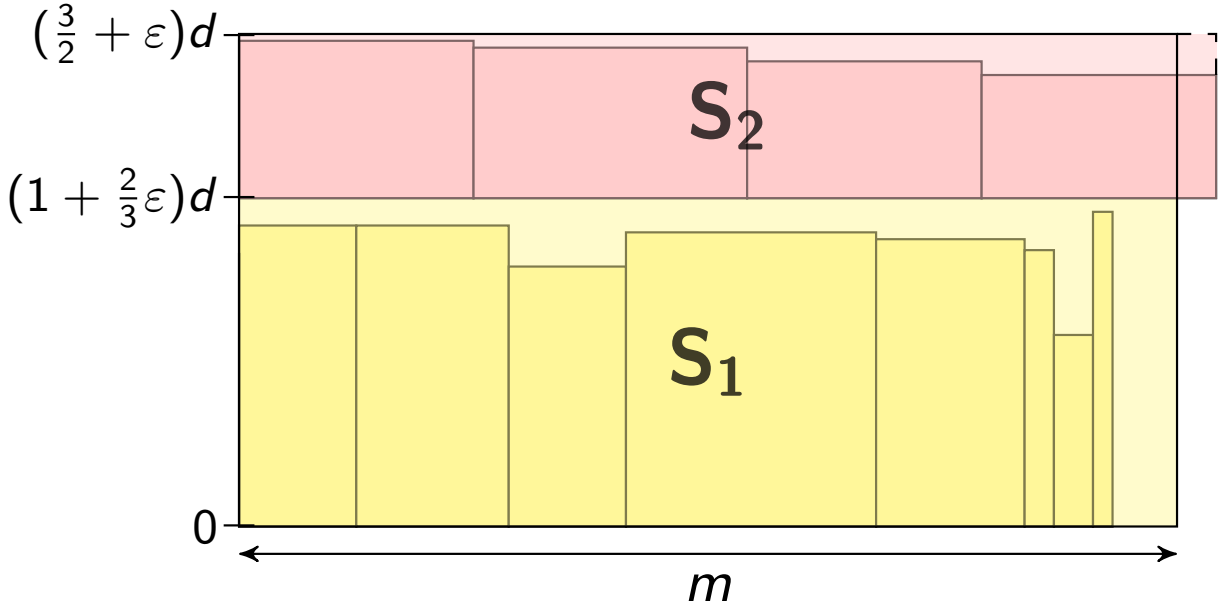

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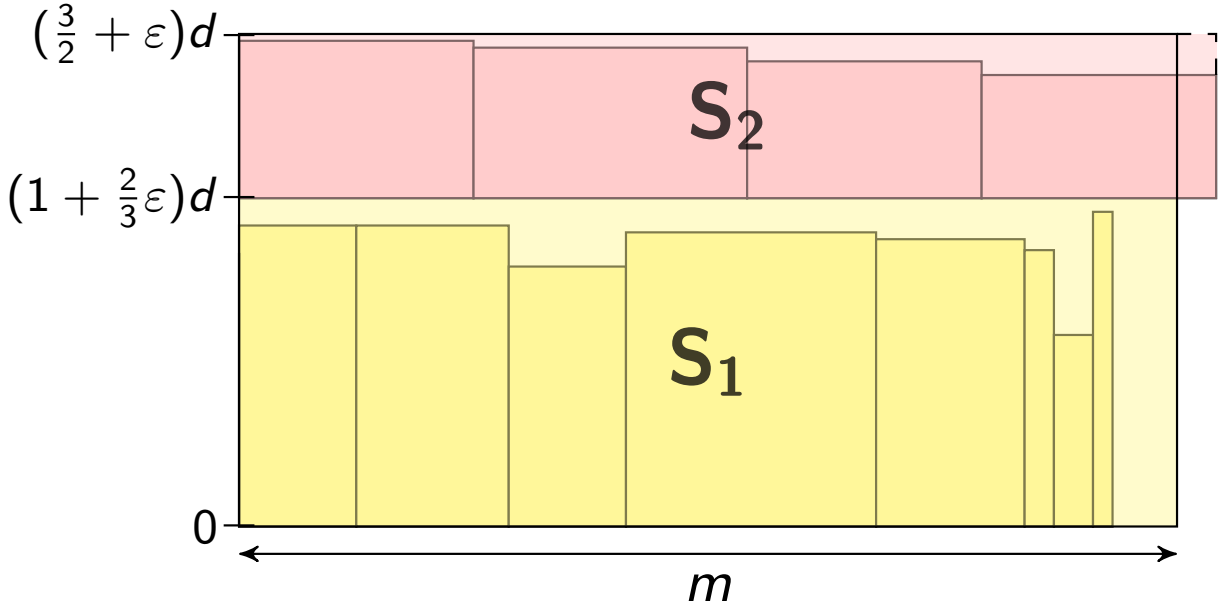

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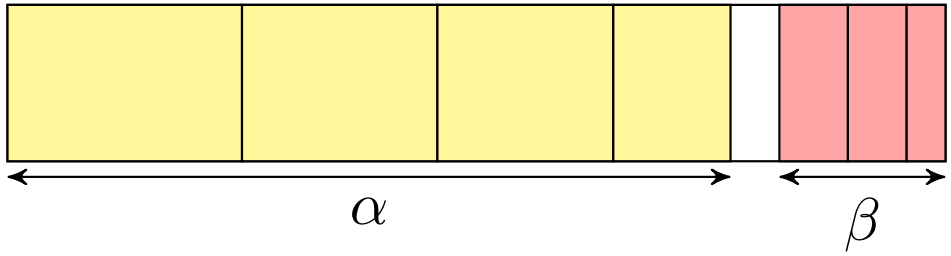
- Increase shelf height – compressed jobs fit, allows for work slightly above md
- Work does not increase – repair procedure works

Solve Knapsack with Compressible Items

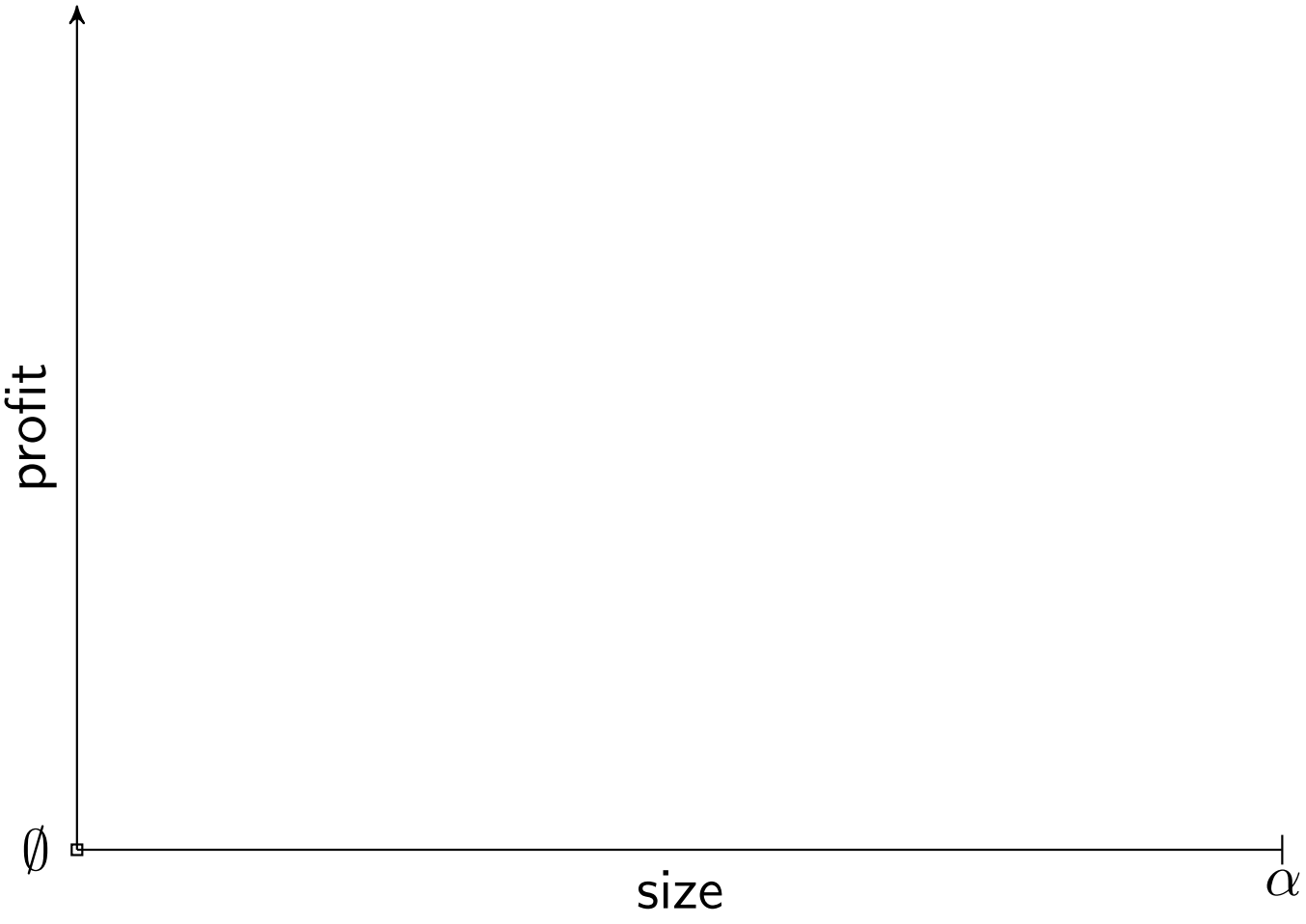


- Handle compressible and incompressible items separately
- Guess α, β approximately
- Knapsack problem for incompressible items has bounded capacity
- Use compression to solve knapsack problem for compressible item

Solve Knapsack with Compressible Items

Dynamic programming framework [Lawler, 1979]

1. Iteratively build list of solutions



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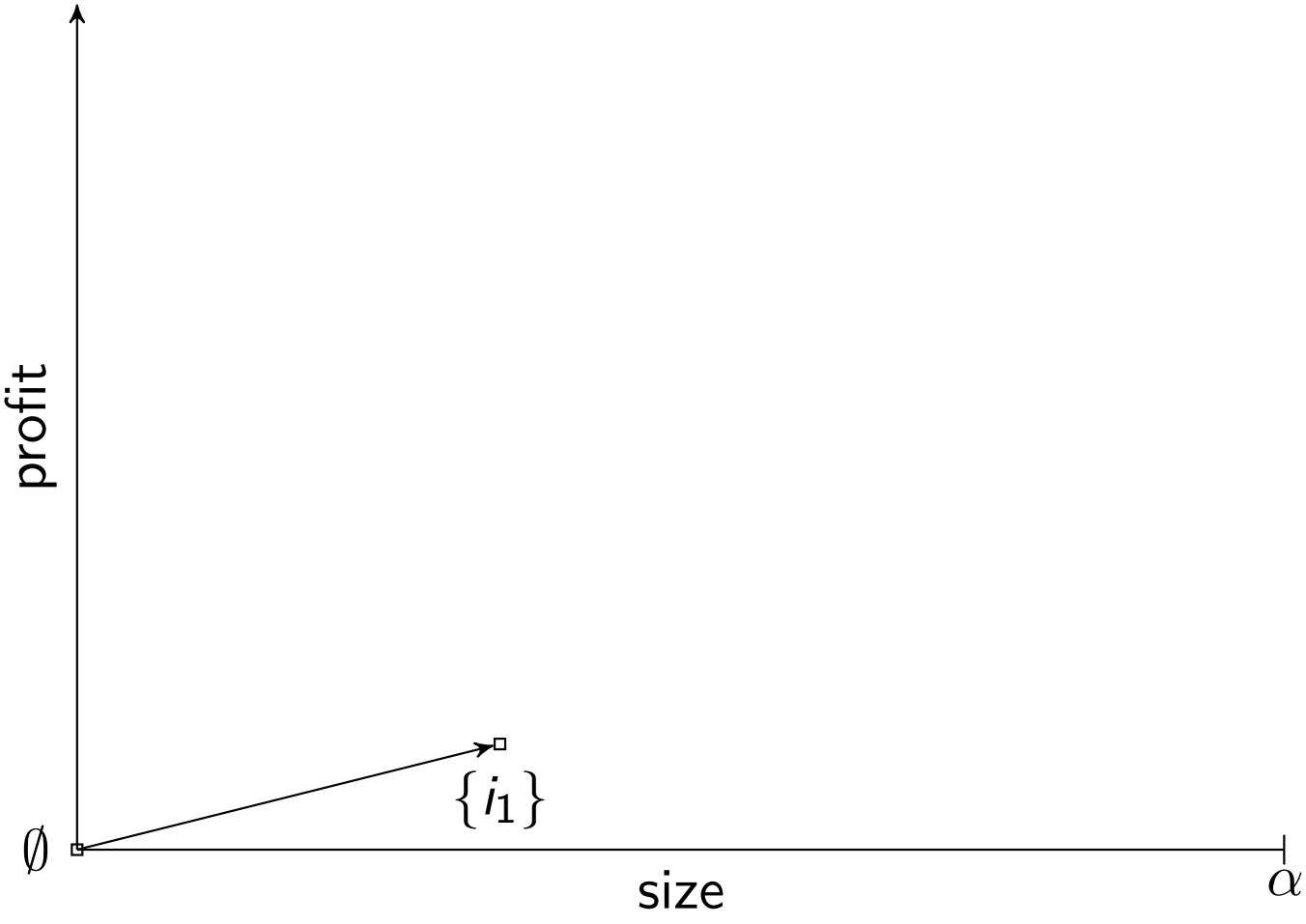

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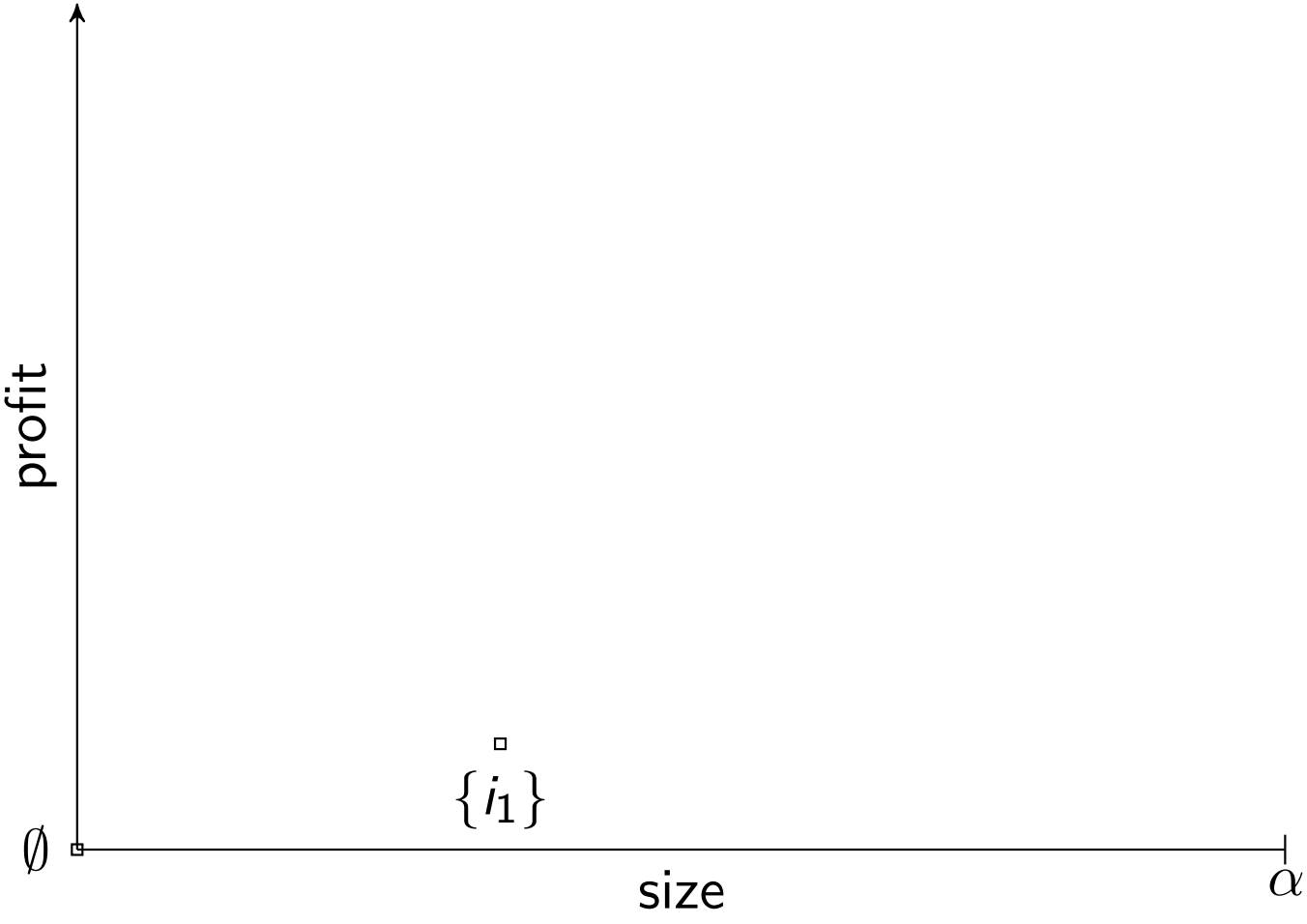

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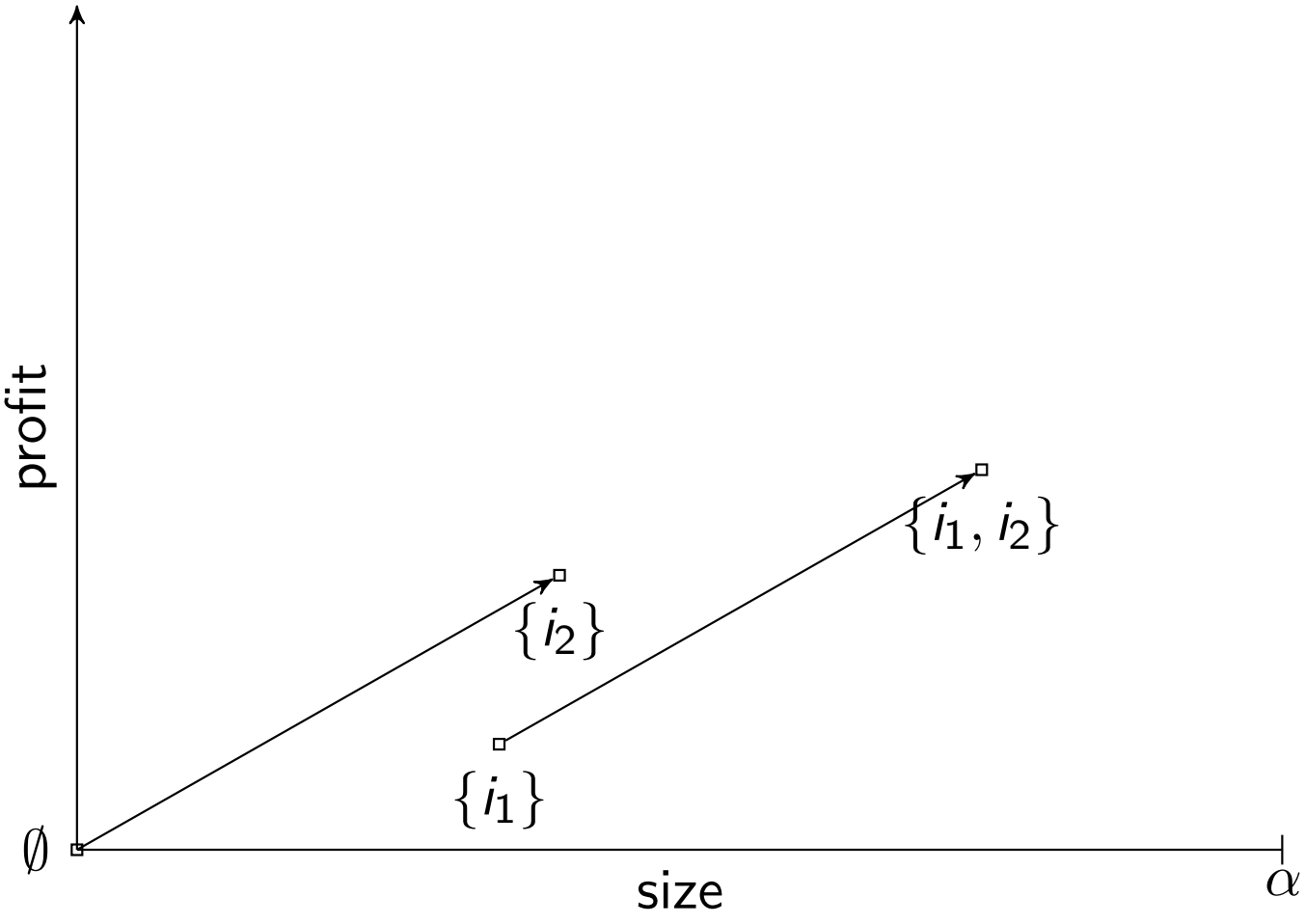

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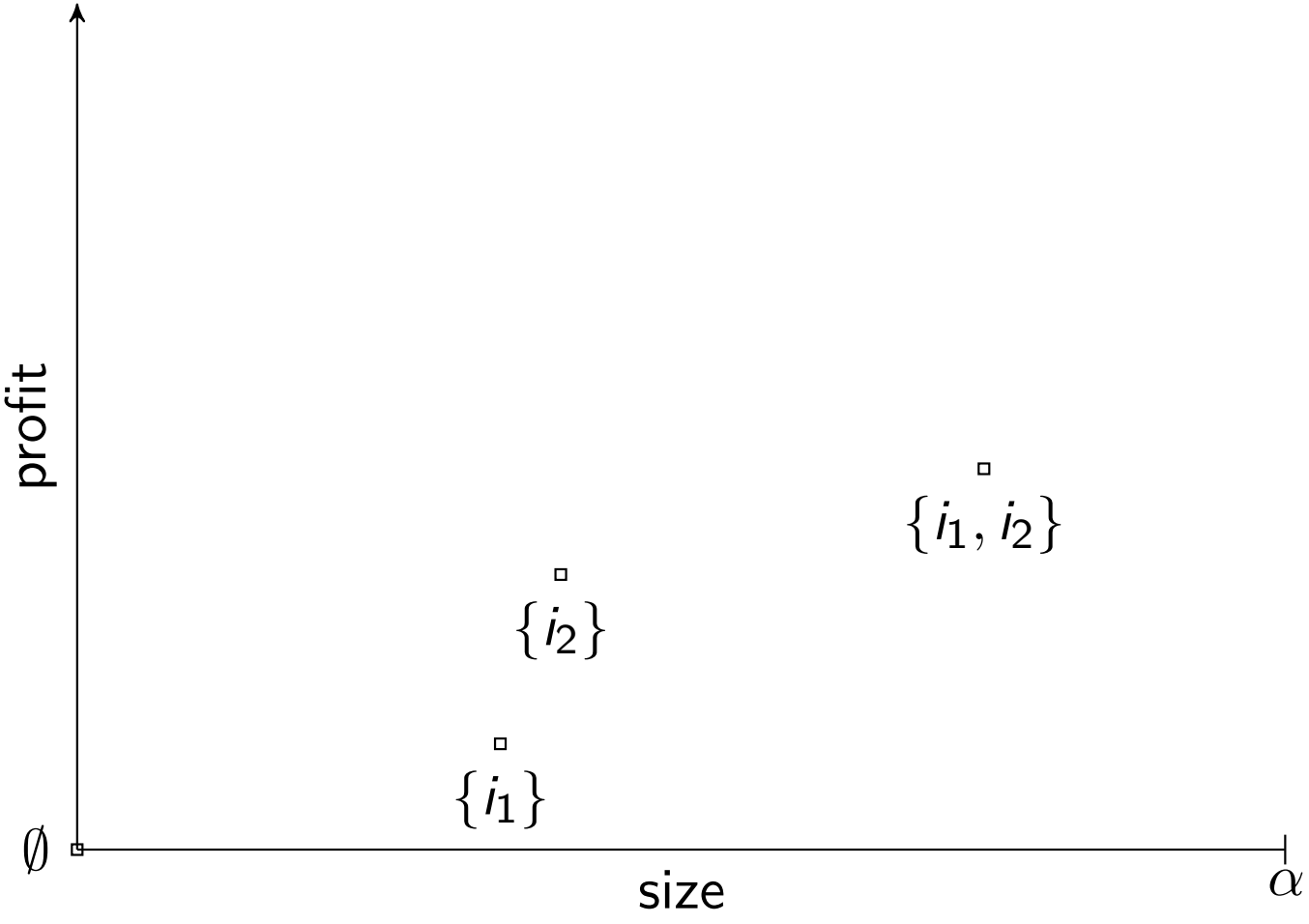
1. Iteratively build list of solutions



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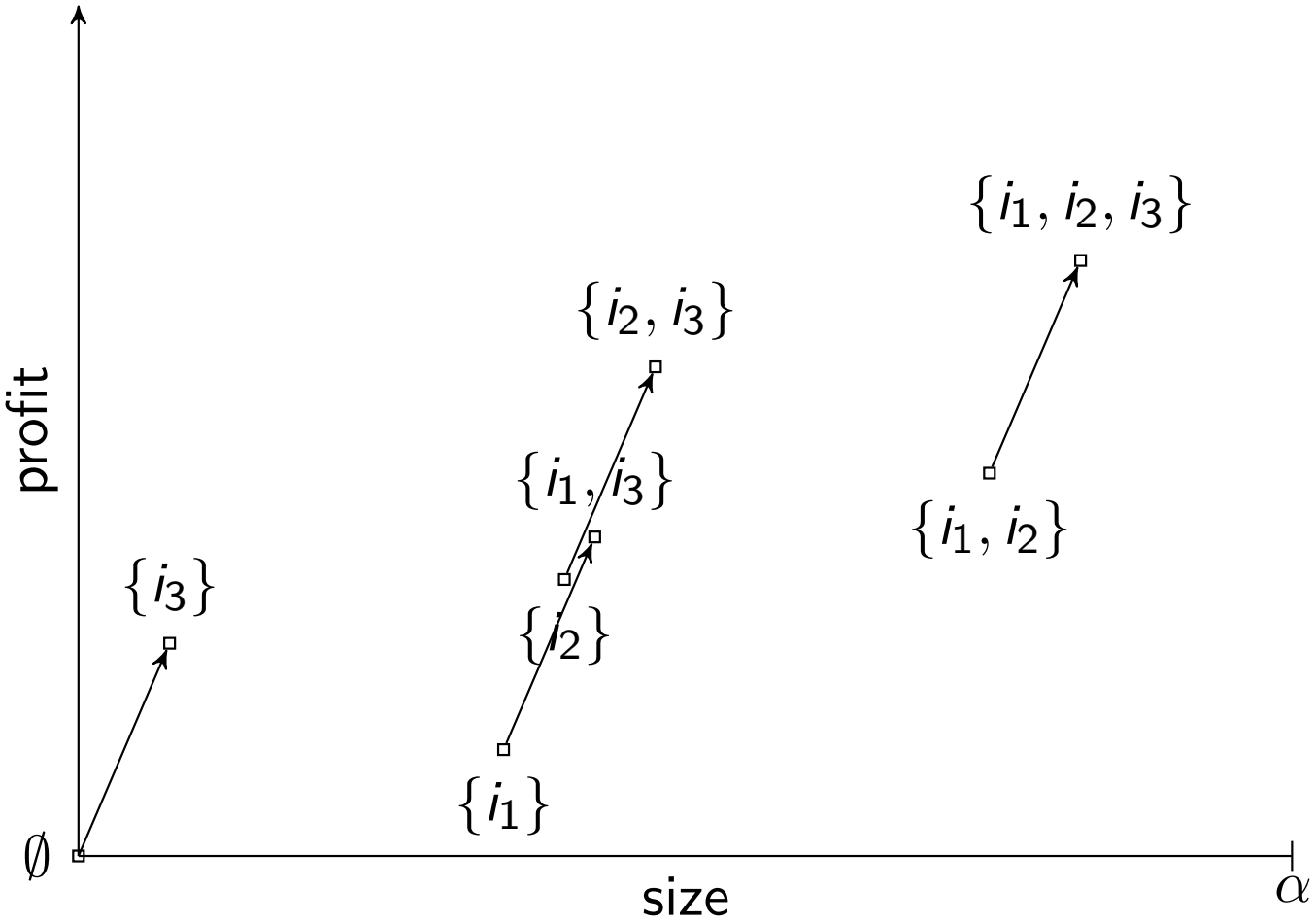

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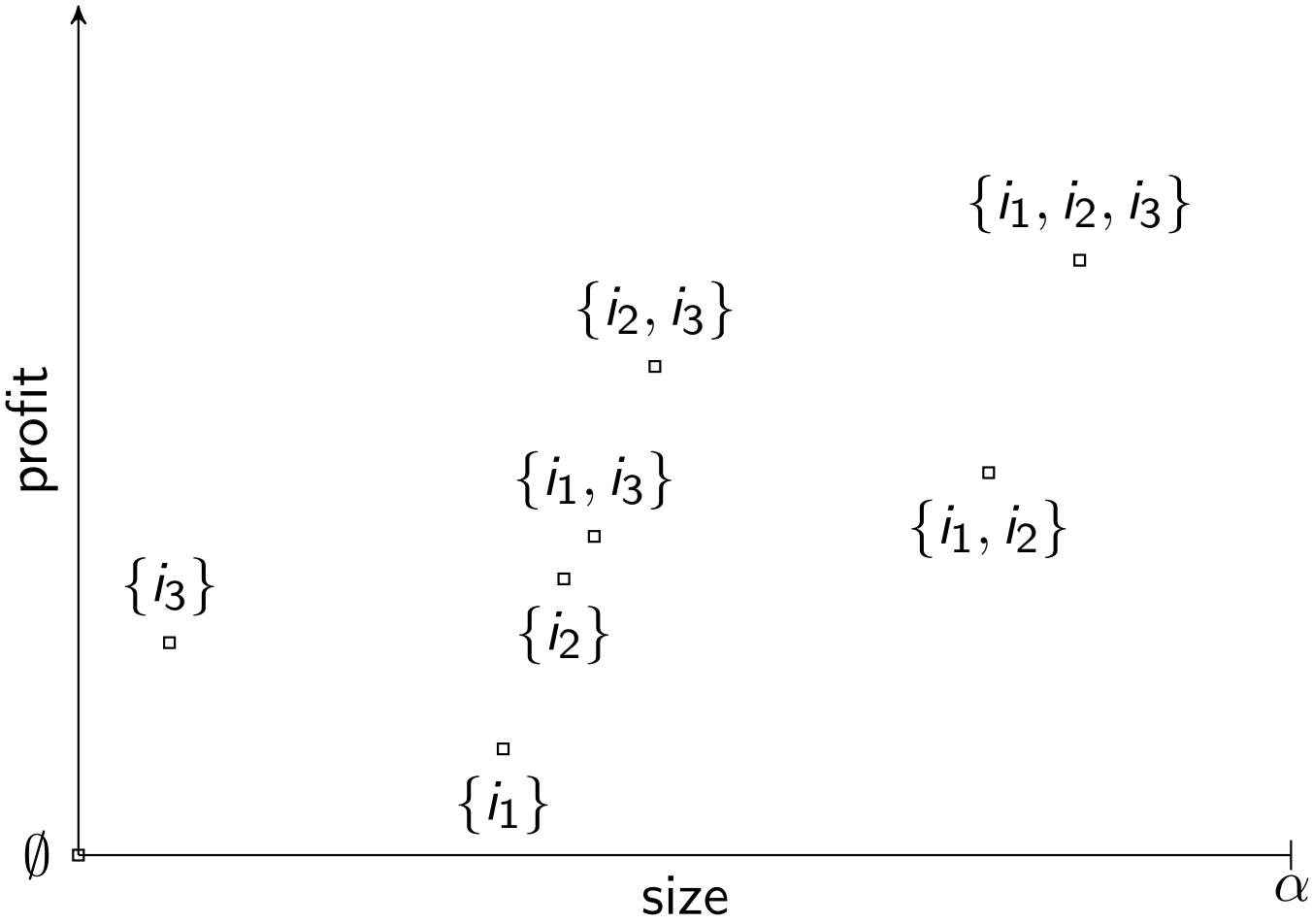
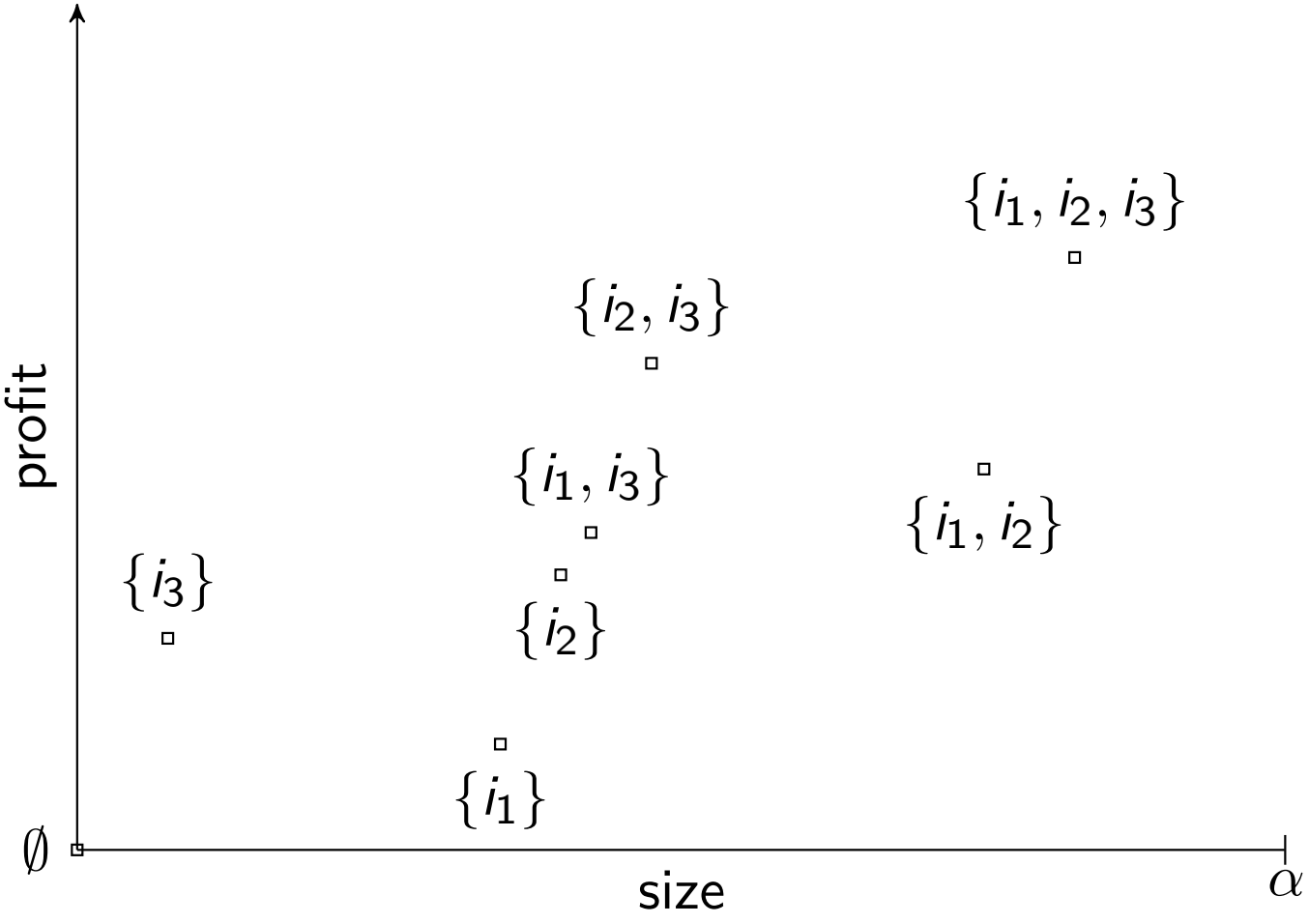


Solve Knapsack with Compressible Items

Dynamic programming framework [Lawler, 1979]

2 Remove dominated solutions



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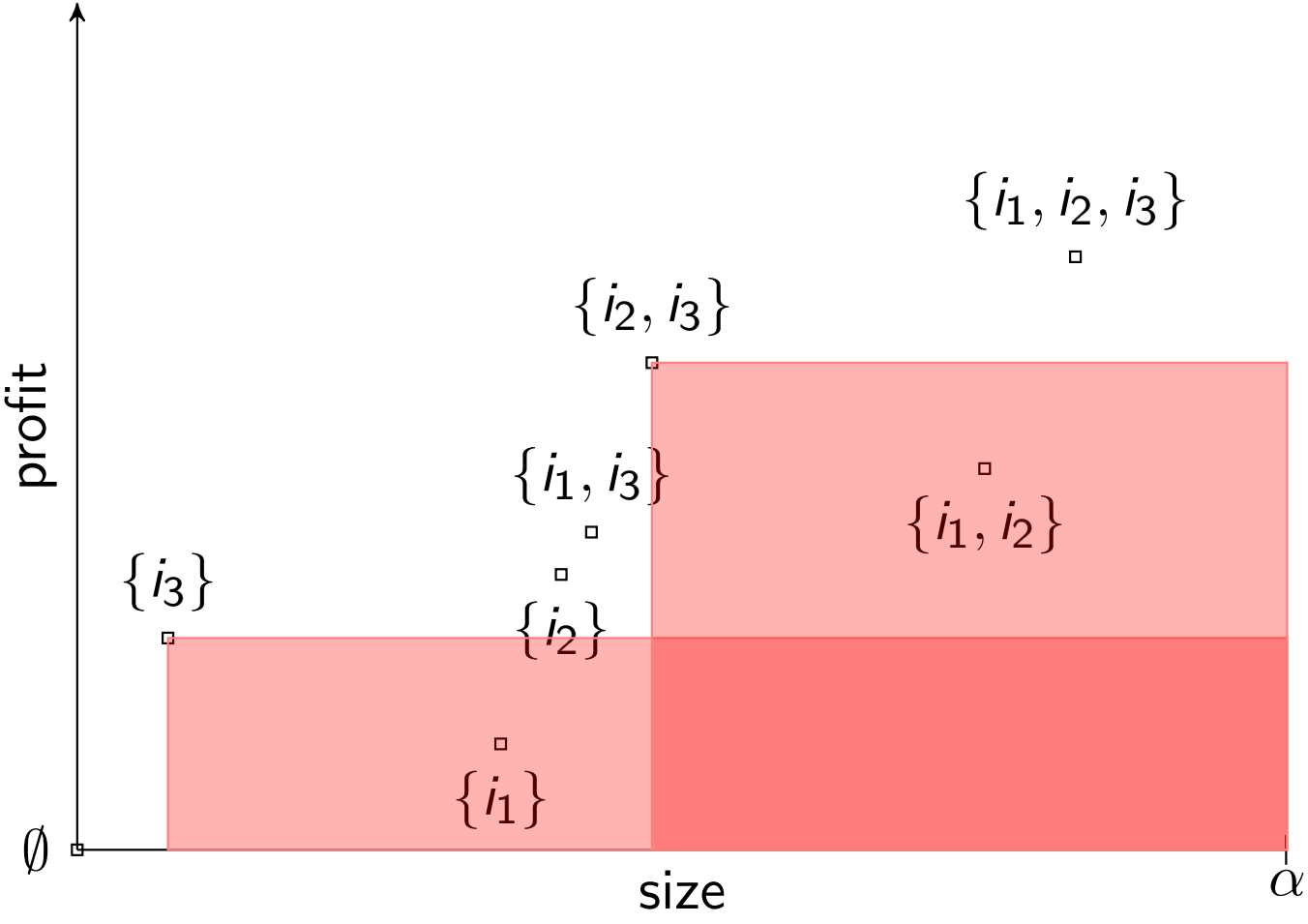

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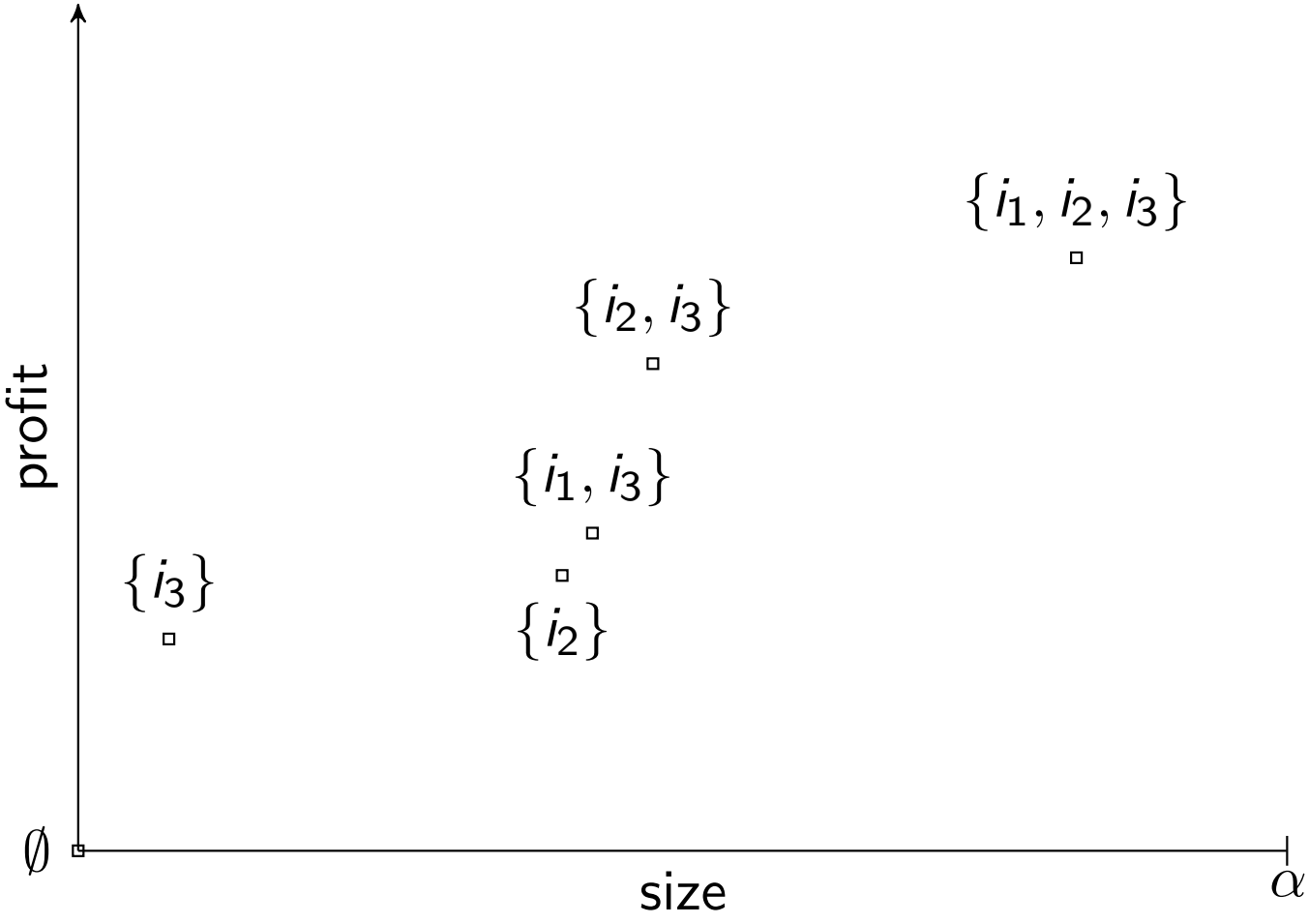

Solve Knapsack with Compressible Items

Dynamic programming framework [Lawler, 1979]

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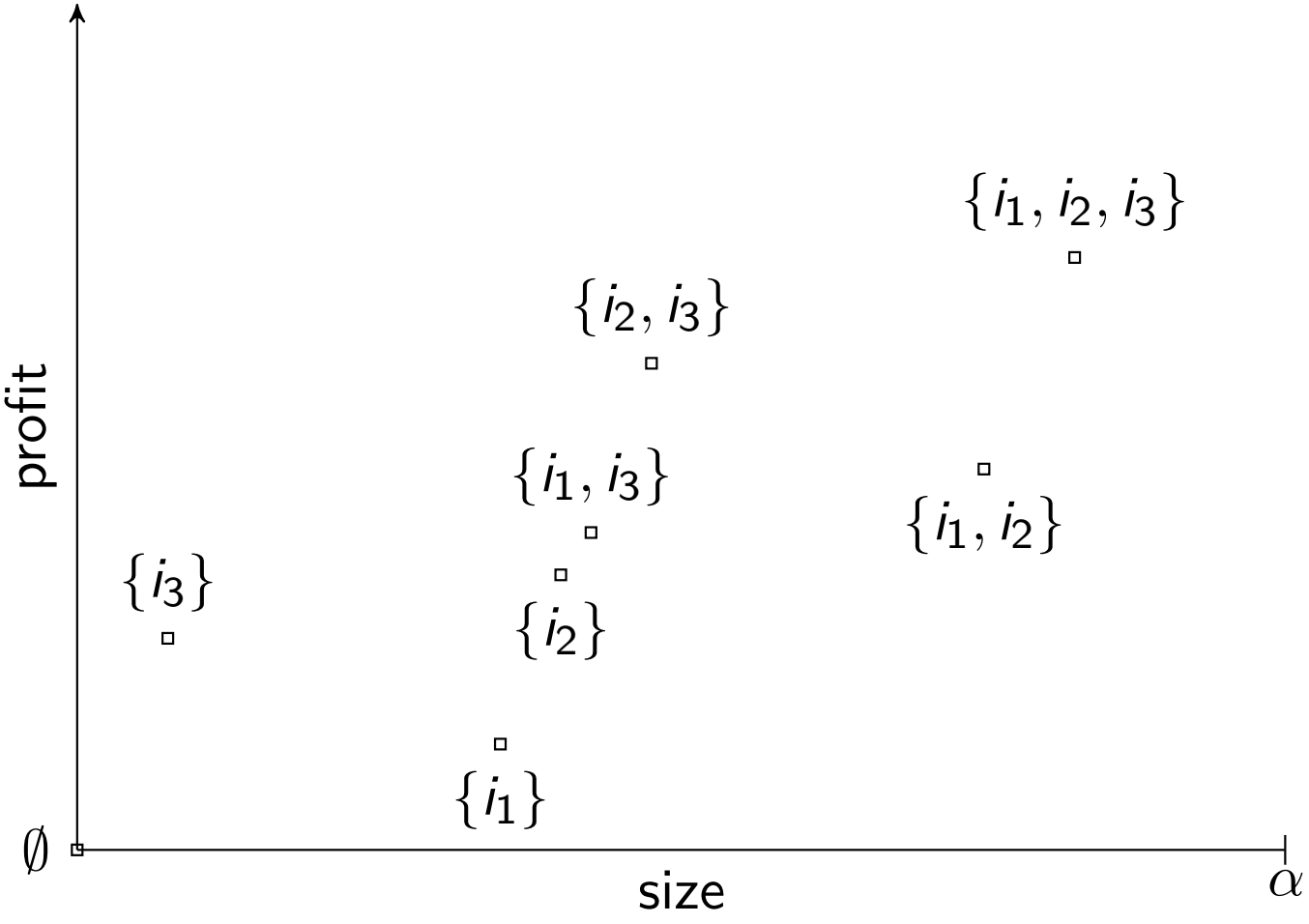

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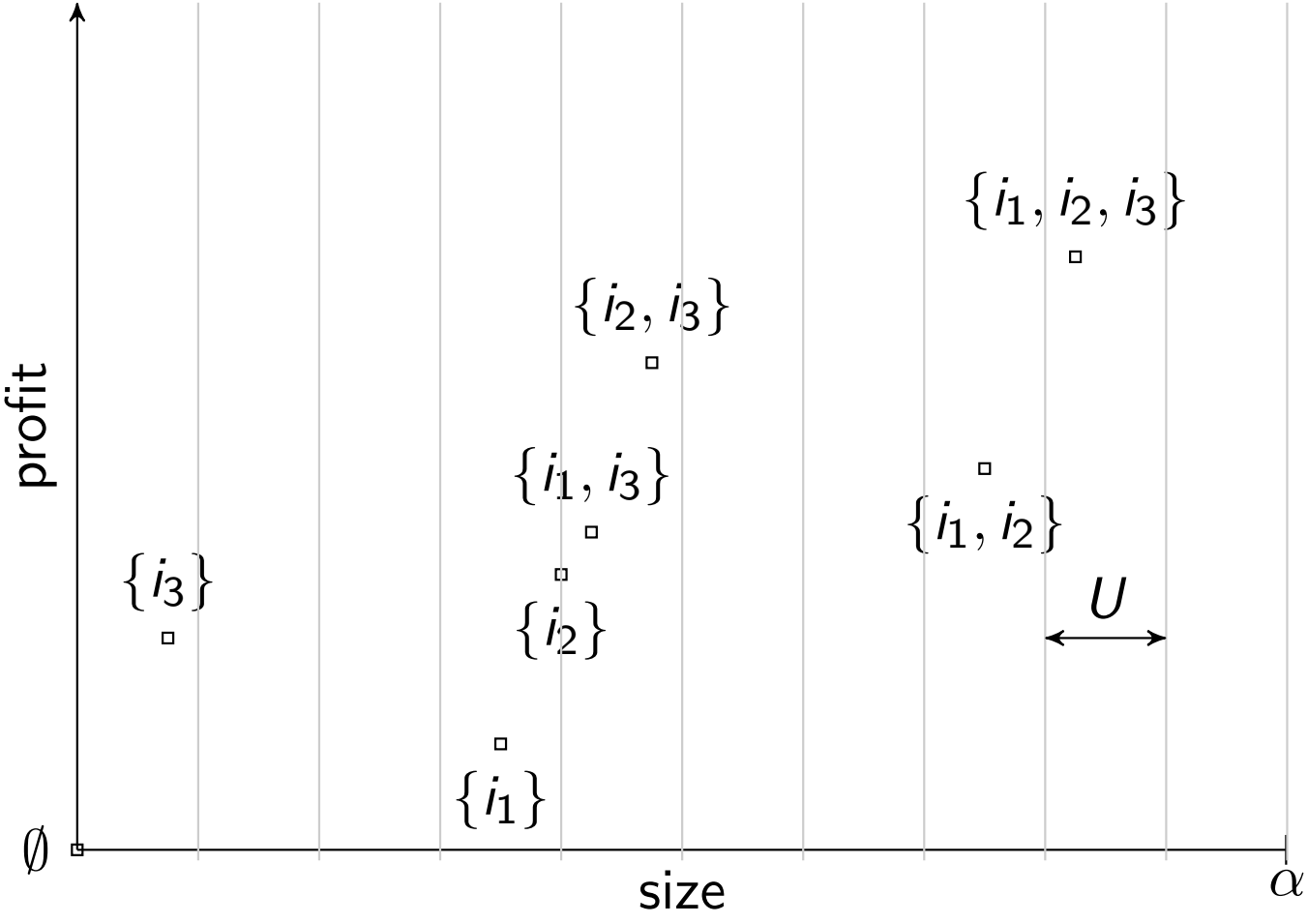

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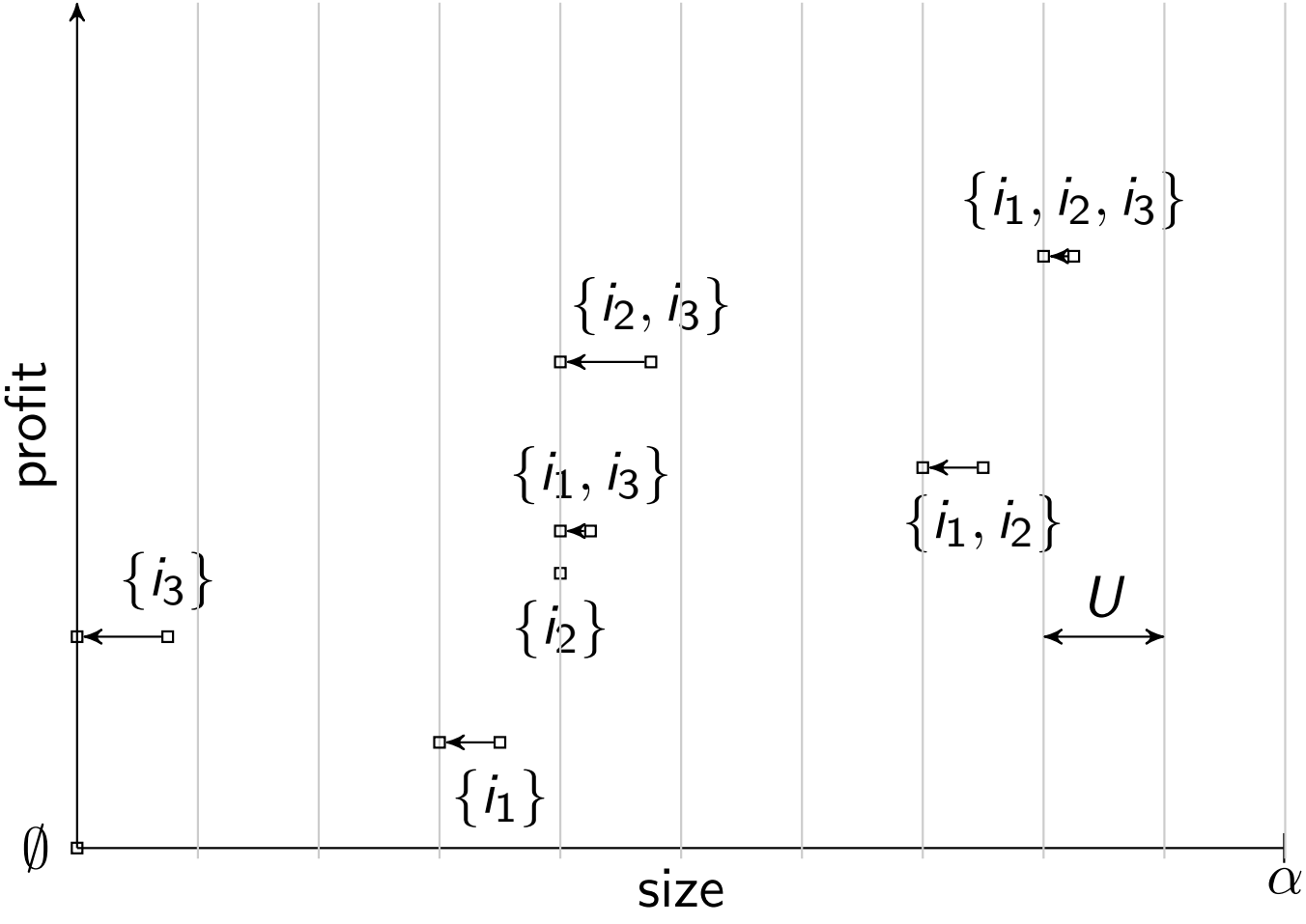

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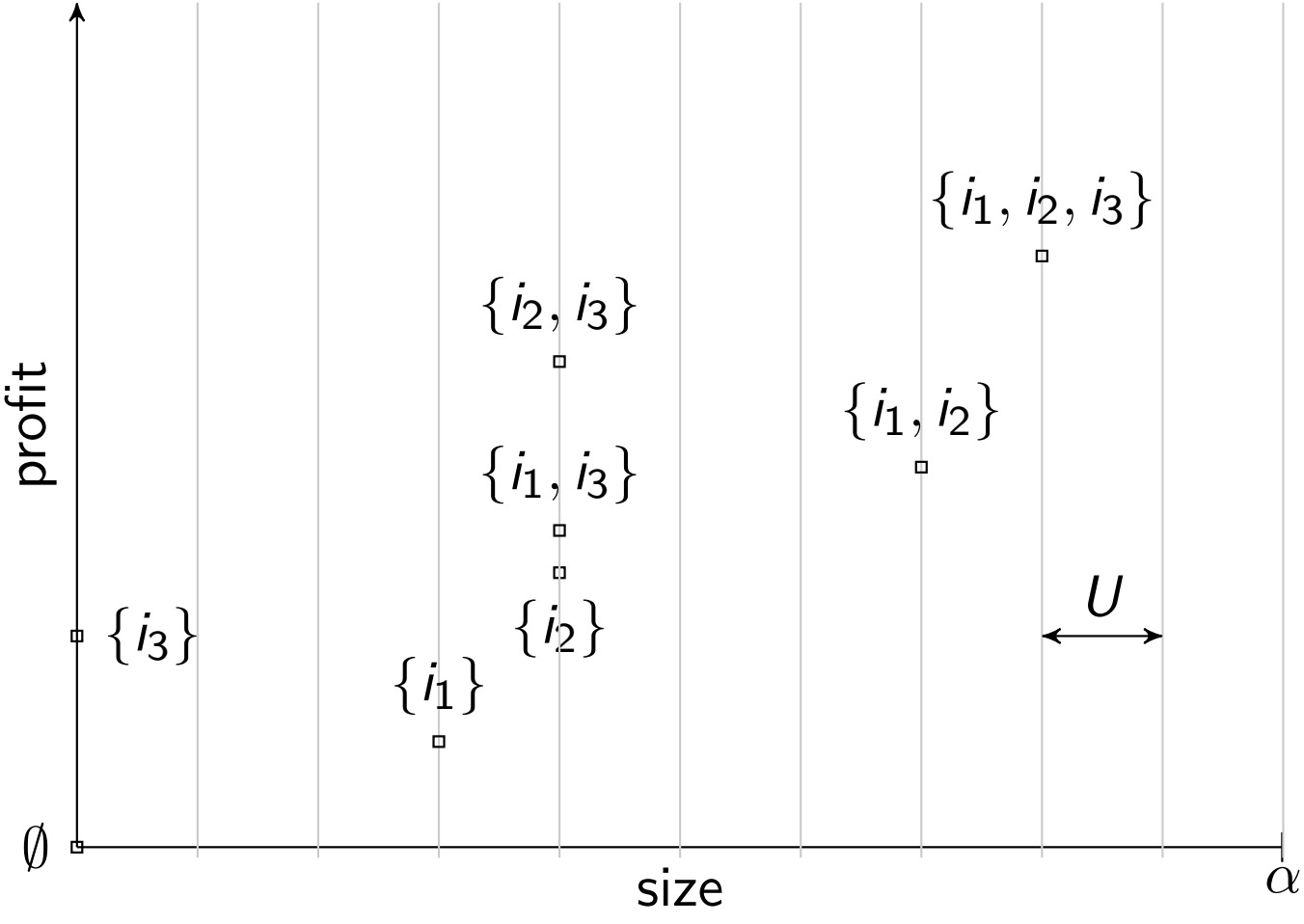

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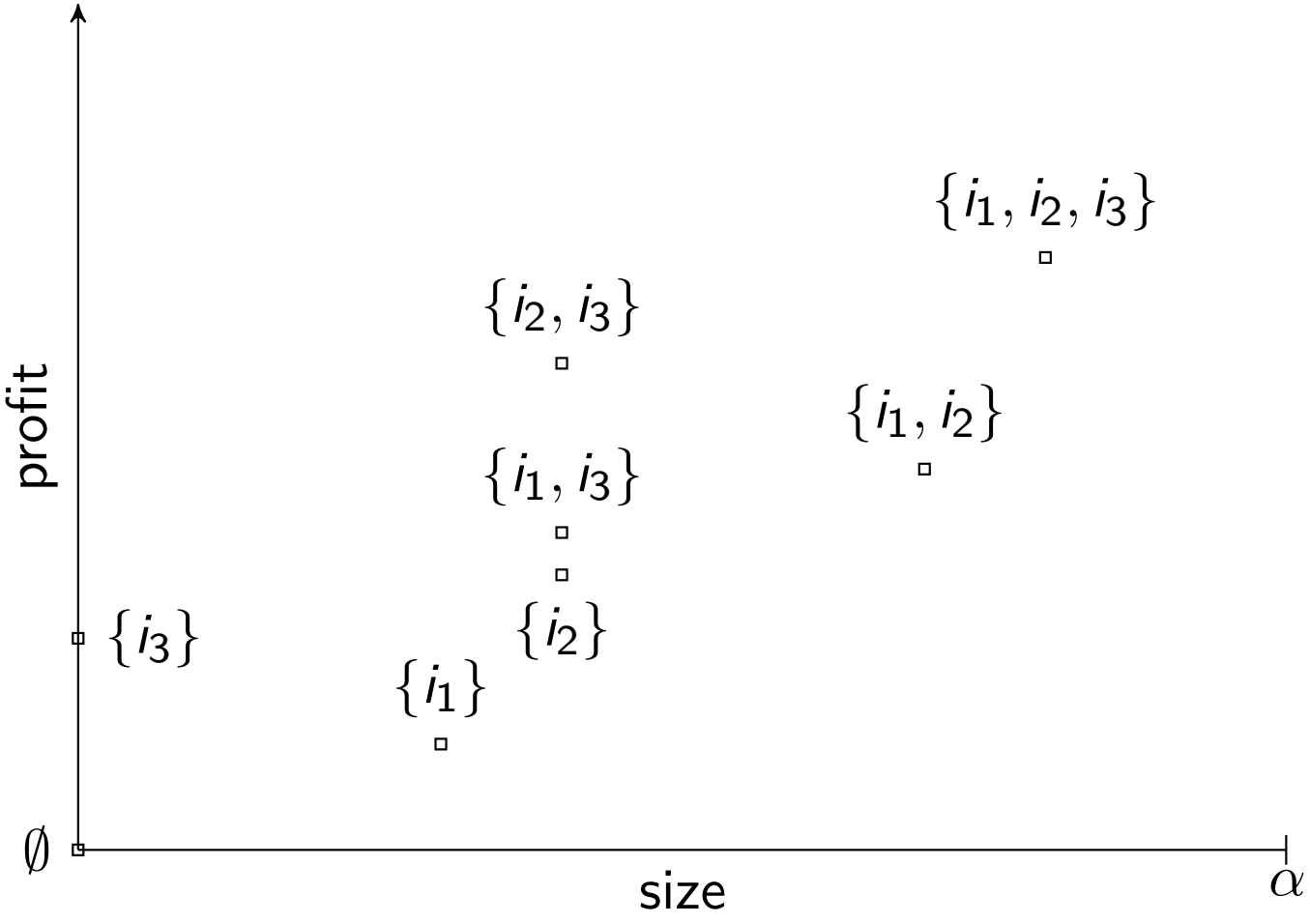

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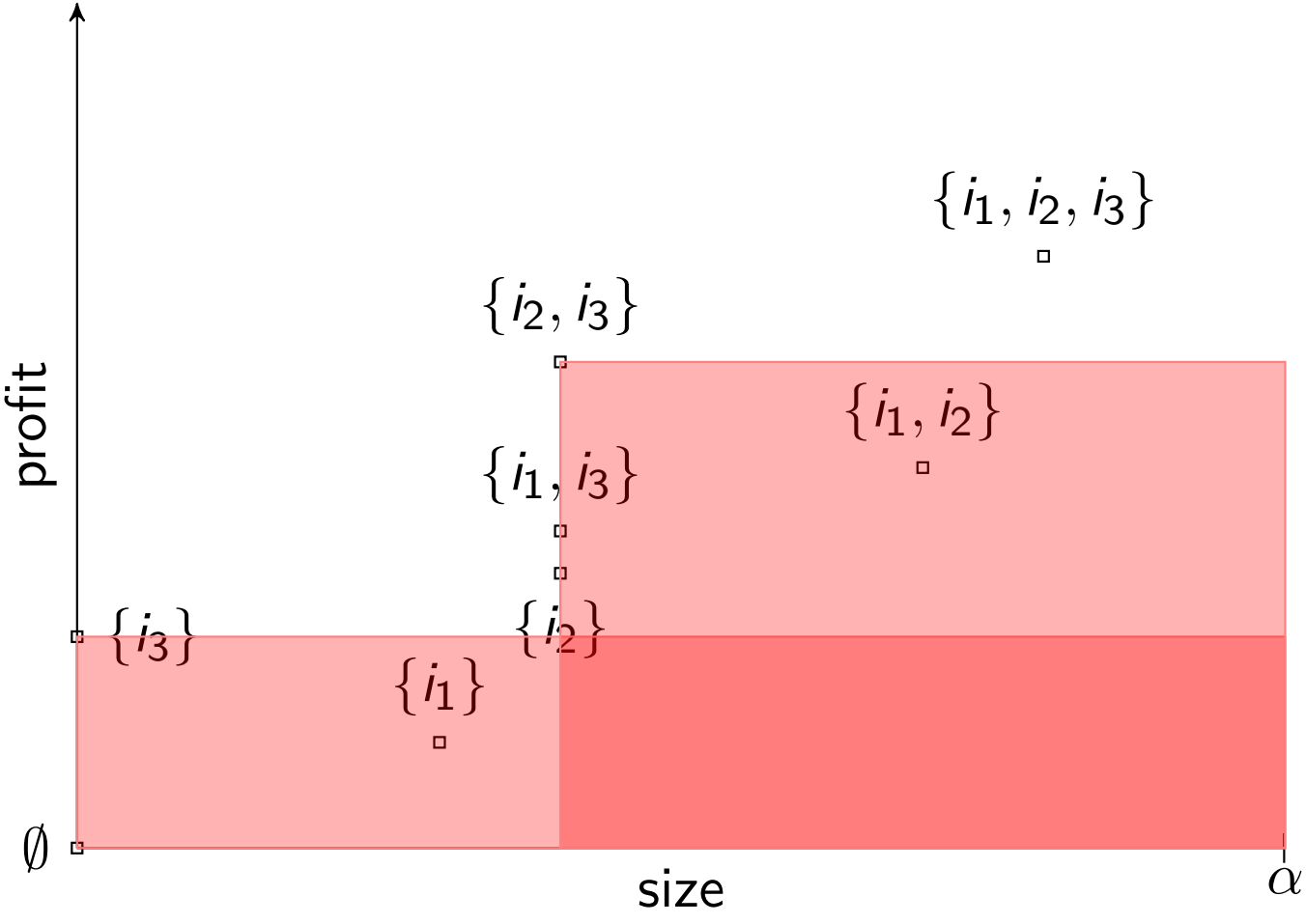

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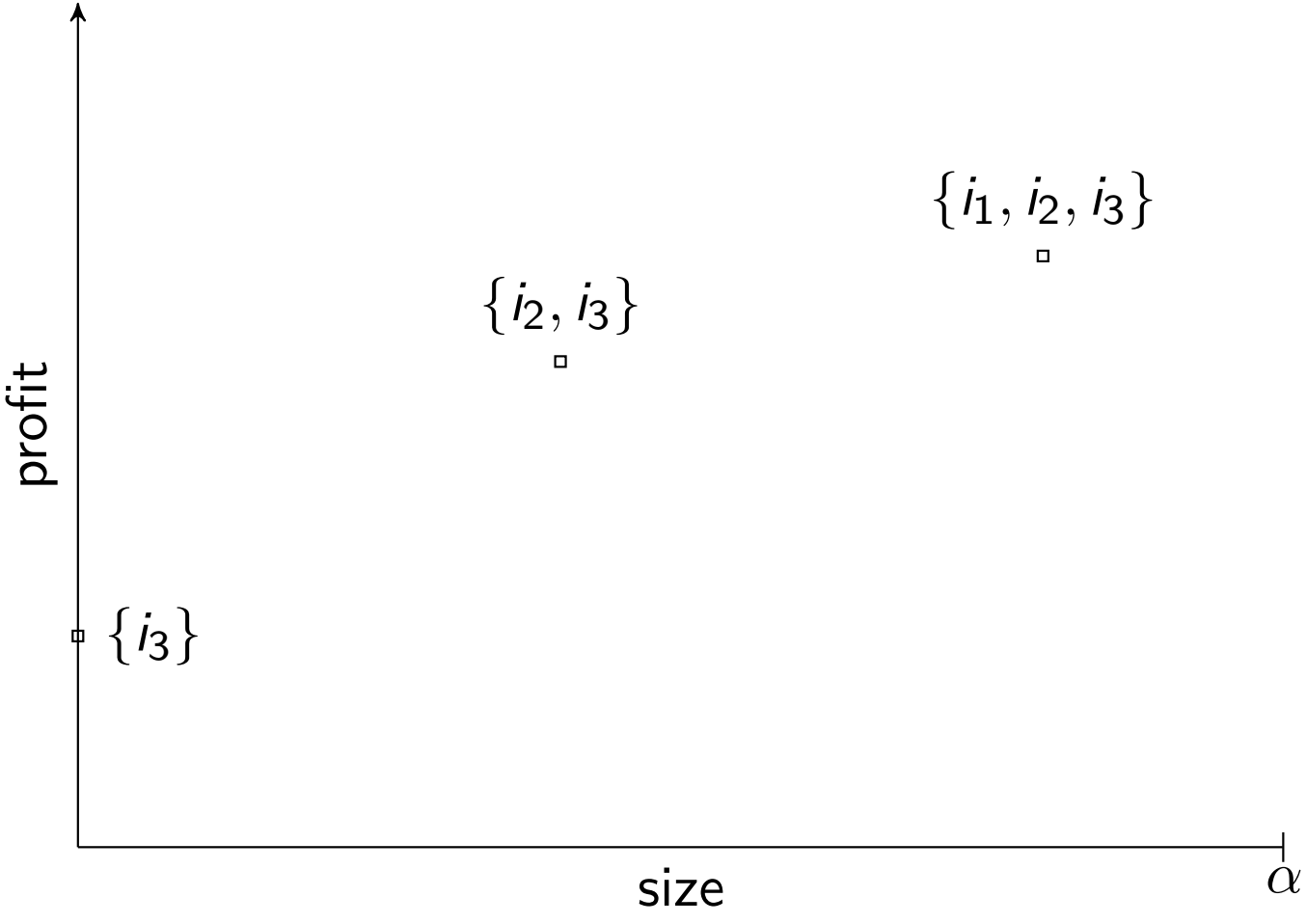

Solve Knapsack with Compressible Items

- Choose $U = \frac{\rho}{(1-\rho)n}\alpha$
- $\lceil \frac{\alpha}{U} \rceil = \mathrm{O}(\frac{n}{\rho})$ many intervals (at most one solution per interval not dominated)
- Solution may use up to $\alpha + nU$ machines
- After compression: uses at most

$$(1-\rho) \times (\alpha + nU) = (1-\rho) \times \left(1 + \frac{\rho}{1-\rho}\right)\alpha = \alpha$$

machines

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  machines

- Running time $\mathrm{O}(n^3)$
- Even more techniques in IPDPS 2018 paper,
- Running time $\mathrm{O}(\frac{n}{\varepsilon^2} \log m \, (\frac{\log(m)}{\varepsilon} + \log^3(\varepsilon m)))$

Improvement (Euro-par 2023)

- Define $S_\rho = [b] \cup \{\lfloor (1+\rho)^i b \rfloor : i \in [\lceil \log_{1+\rho}(m/b) \rceil]$ for $\rho = \varepsilon/4$ and $b = 1/\rho$.
- Then $|S_\rho| = O(\frac{1}{\varepsilon} + \frac{\log(\varepsilon m)}{\varepsilon})$.

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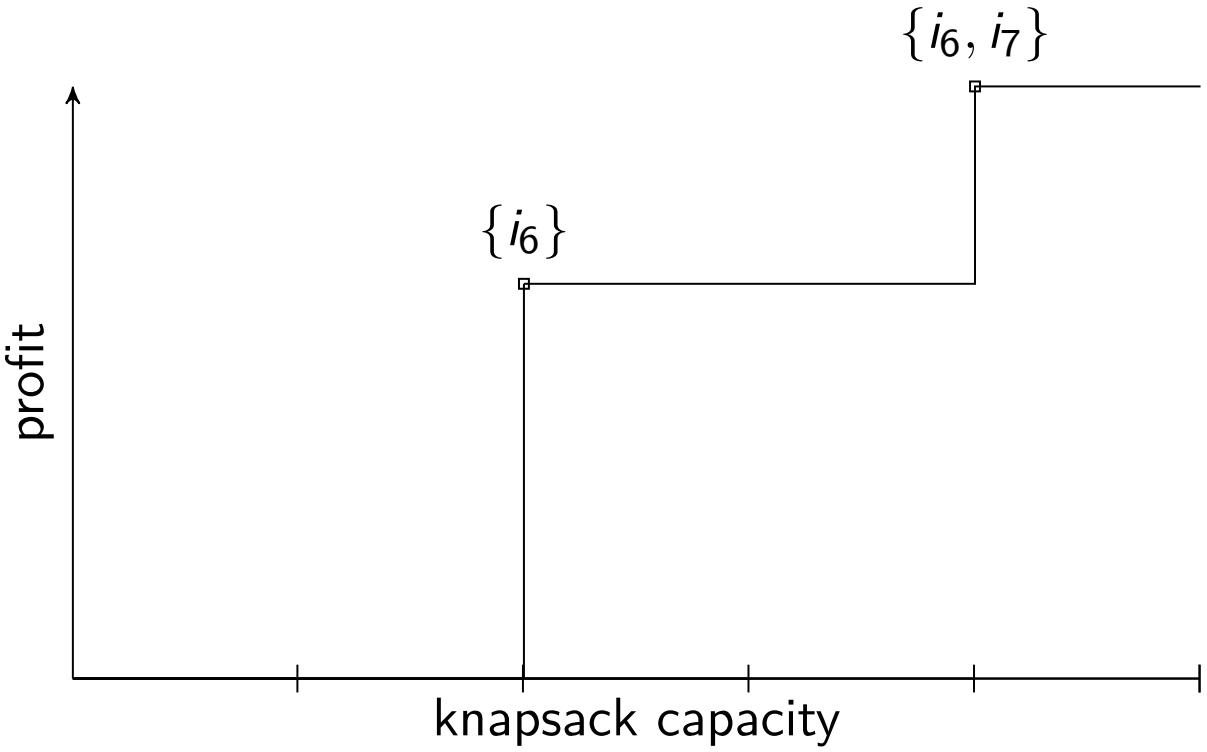
- Reduce machine counts to

$$\gamma'(j,s) = \max\{k \in S_\rho : k \leq \gamma(j,s)\}$$

for $s \in \{d/2, d\}$.

Solving Knapsack (Axiotis and Tzamos)

1 Construct a solution sequence R_i for each item weight $w_i \in S_\rho$ separately



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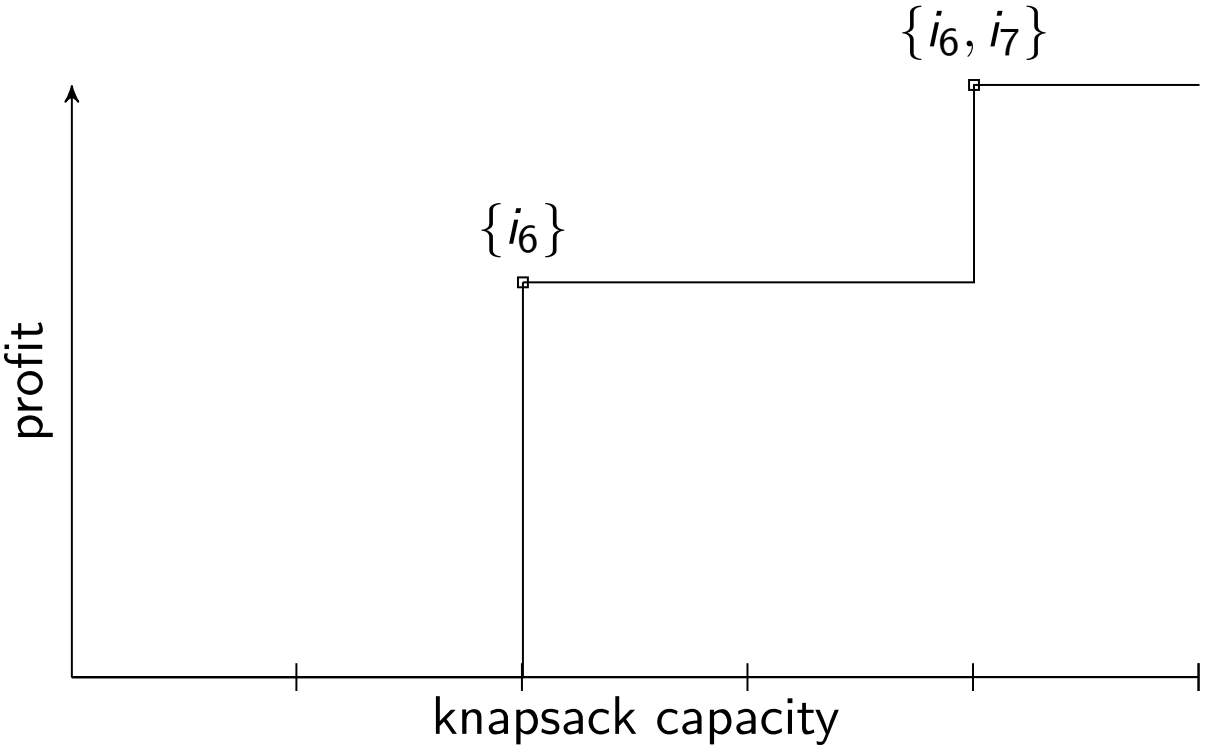
   1.1 $w(i_6) = w(i_7) \in S_\rho$



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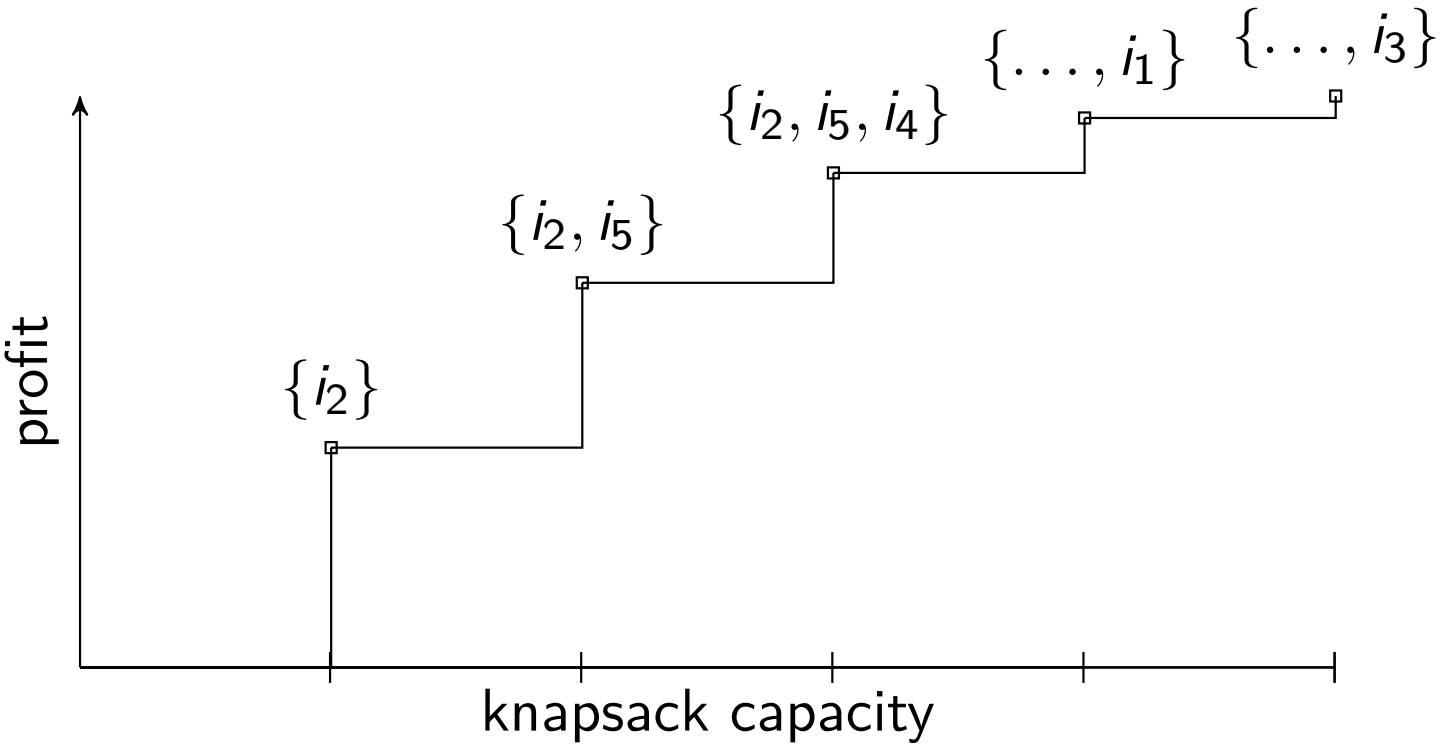
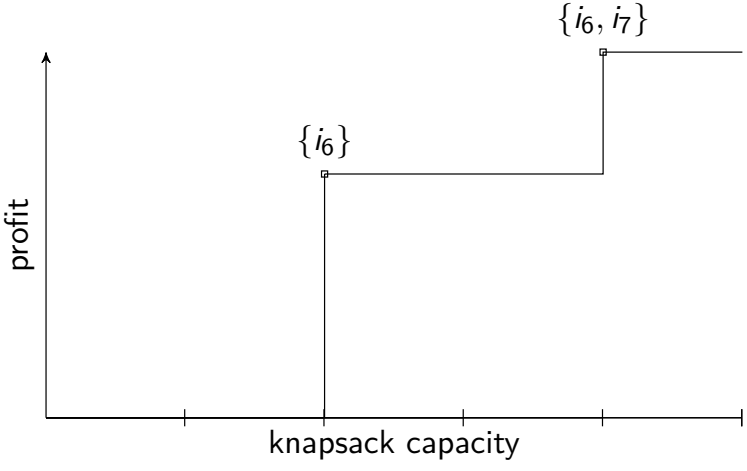
1 Construct a solution sequence R_i for each item weight $w_i \in S_\rho$ separately
   1.2 $w(i_1) = w(i_2) = w(i_3) = w(i_4) = w(i_5) \in S_\rho$

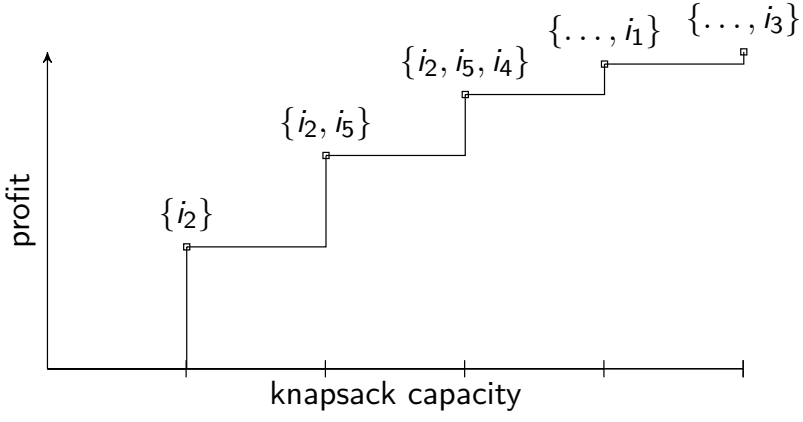


Solving Knapsack (Axiotis and Tzamos)

2 Combine $(((R_1 \oplus R_2) \oplus ...) \oplus R_{|S_\rho|})$ via *(max,+)-convolution*

- efficient for *k-step concave* sequences





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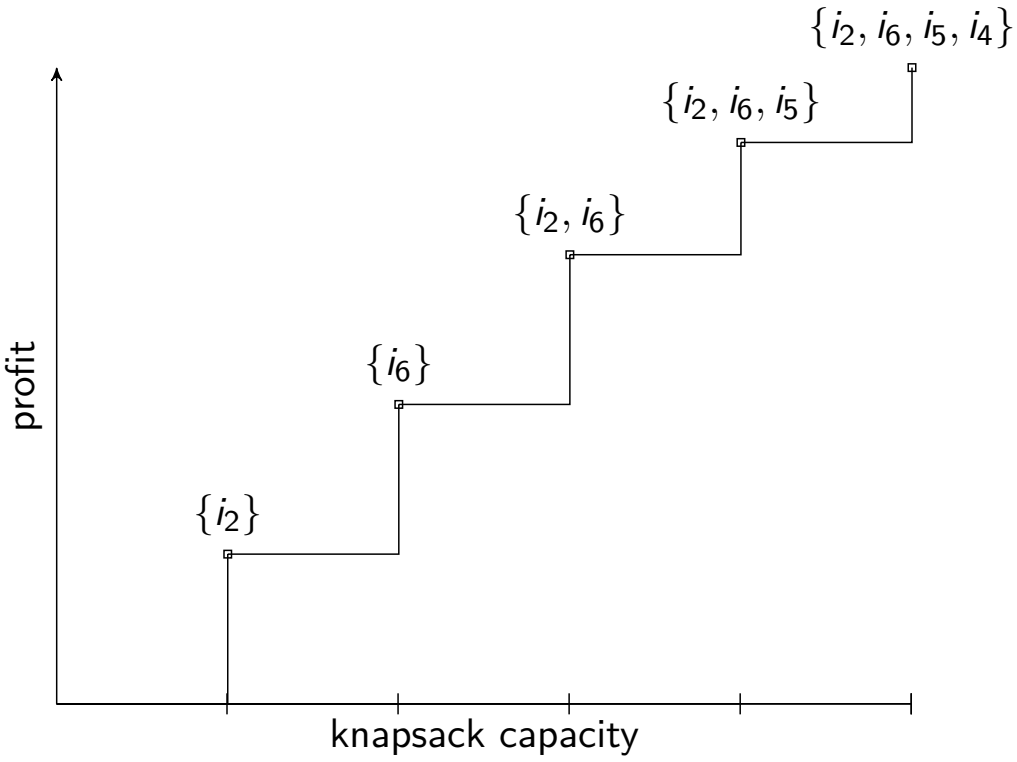
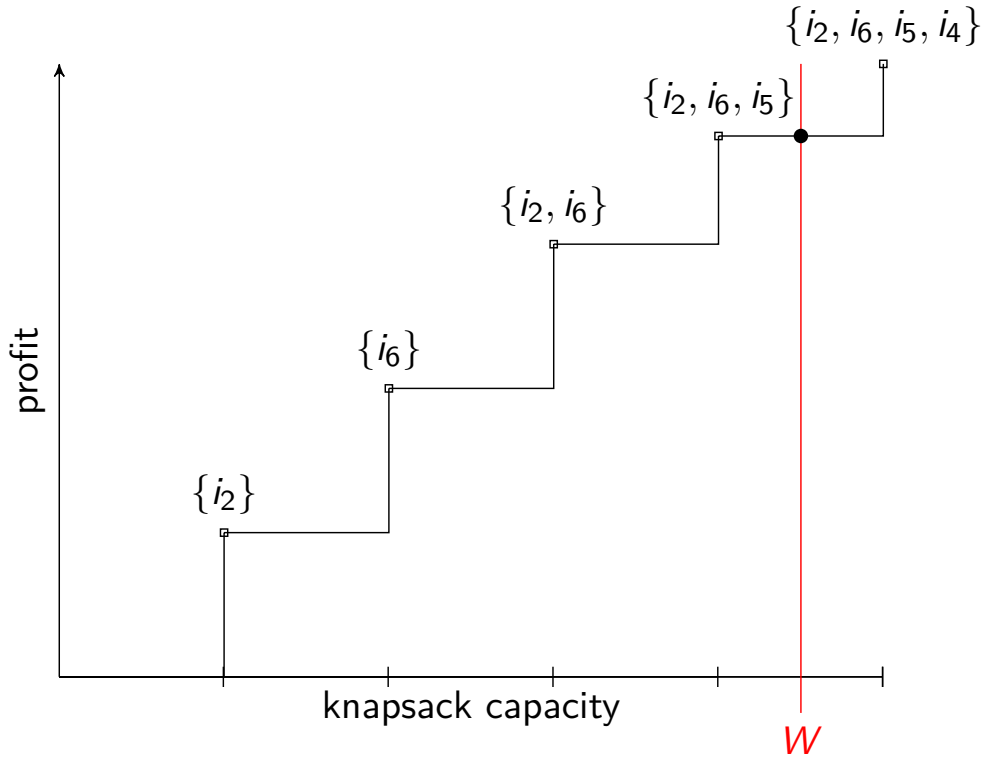


Solving Knapsack (Axiotis and Tzamos)

3 Final solution is given in by entry at capacity W



Improvement (Euro-par 2023)

- Distinguish between wide/narrow jobs in S_1 and S_2:
  - Jobs narrow in both shelfs: profit is reduced to next multiple of εd.
  - Jobs wide in both shelfs: modify processing time to $\frac{s}{1+4\rho}$, work to $\frac{s}{1+4\rho}\gamma'(j,s)$ for $s \in \{d/2, d\}$.

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  - Jobs narrow in one shelf and wide in the other: $\gamma'(j,d/2) \geq b > \gamma'(j,d)$
    - wide part: use processing time $\frac{1}{1+4\rho}\frac{d}{2}$ and work $w'(j,d/2) = \frac{1}{1+4\rho}\frac{d}{2}\dot{\gamma}'(j,d/2)$
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    - narrow part: round down the work to next multiple of εd
- Use knapsack algorithm by Axiotis and Tzamos (partition items into sets of the same weight, solve separately for each weight and combine solutions with $(\max,+)$-convolution).
- Running time $\mathrm{O}(n\log^2(\frac{1}{\varepsilon}+\frac{\log(\varepsilon m)}{\varepsilon})+\frac{n}{\varepsilon}\log(\frac{1}{\varepsilon})\log(\varepsilon m))$

Conclusion

- Resolved complexity for monotone jobs
  - Previous: Weakly NP-hardness
  - New: Strongly NP-hardness
- Resolved approximability for monotone jobs
  - Previous: $(\frac{3}{2} + \varepsilon)$ and $\frac{3}{2}$-approximation
  - New: Polynomial time approximation scheme (PTAS)
- Efficient algorithms for monotone jobs
  - New: FPTAS when $m \geq \frac{8}{\varepsilon} n$
  - New: $(\frac{3}{2} + \varepsilon)$-approximation: fully polynomial, linear in n/ε

Open Questions

- Can we further reduce running time of our algorithms?
  - via new approximation schemes for knapsack
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  - via new approximation schemes for knapsack
  - via new parameterized algorithms for knapsack
- Can we get a faster PTAS for $m \leq O(n/\varepsilon)$ using monotony?
- Can we achieve more efficient approximation algorithms with ratio $3/2$ or even $< 3/2$?