High-Multiplicity Scheduling on Identical Machines

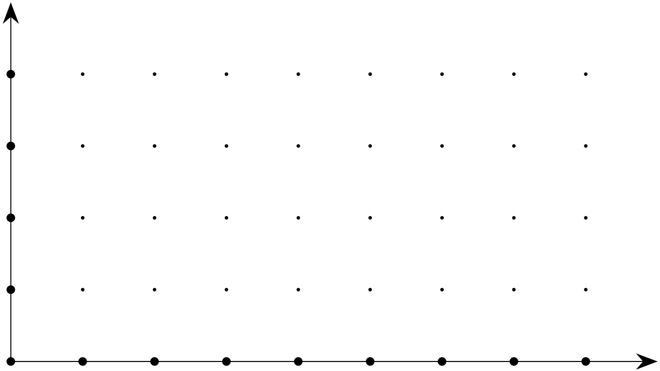
Kai Kahler

joint work with Klaus Jansen and Esther Zwanger

Kiel University

May 16, 2024

ConeAndPolytopeIntersection



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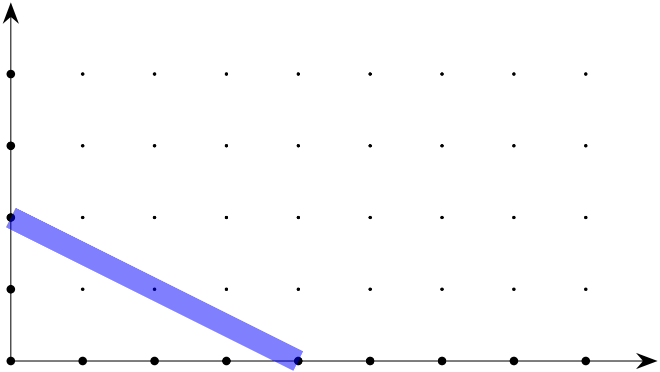

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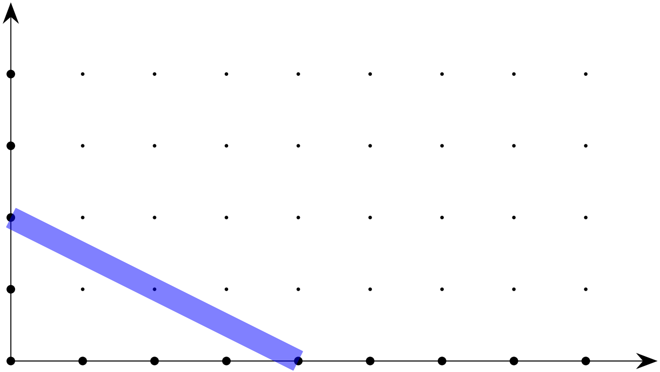

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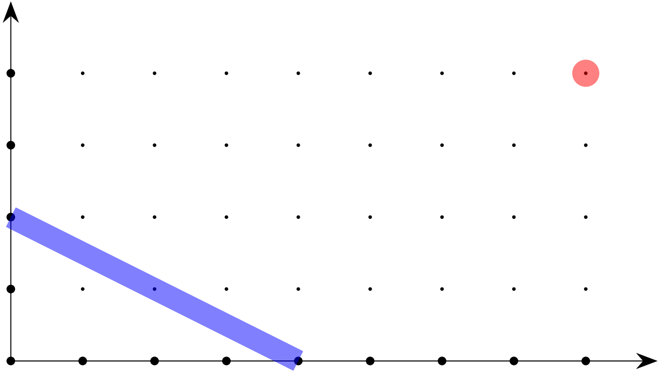

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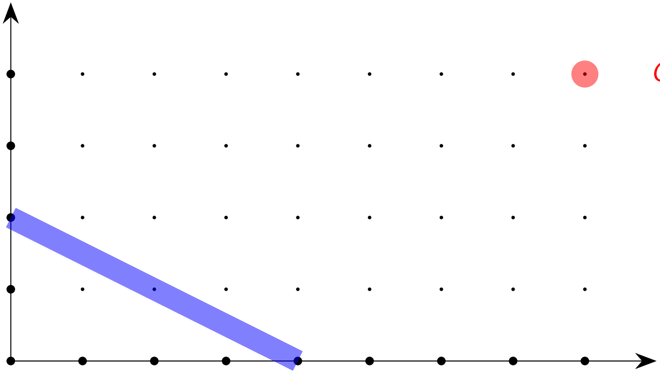

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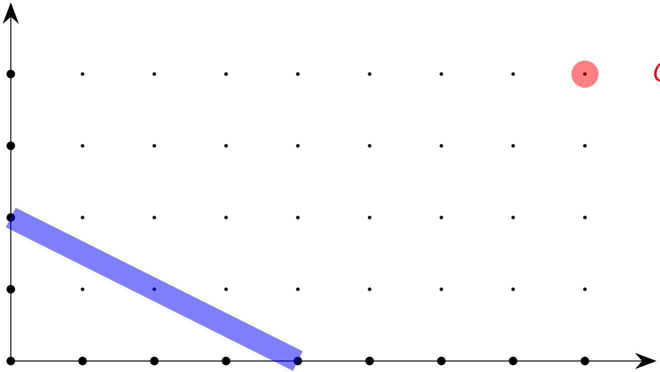

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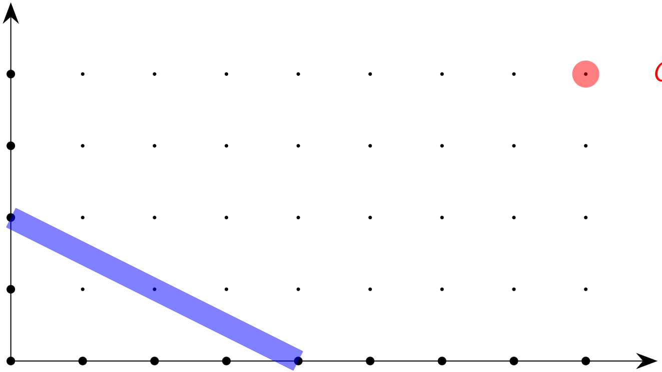

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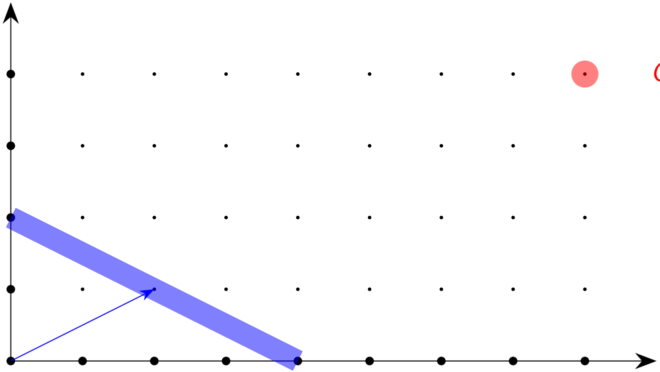

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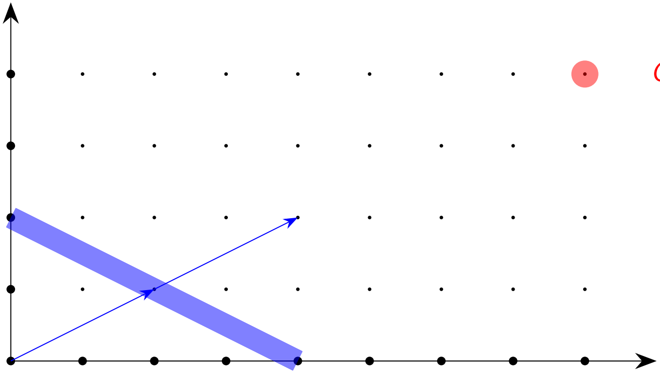

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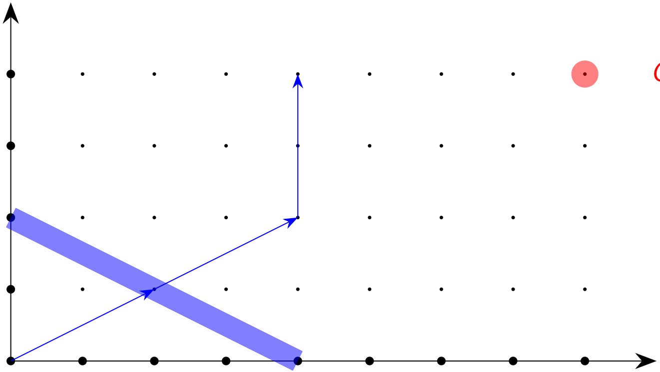

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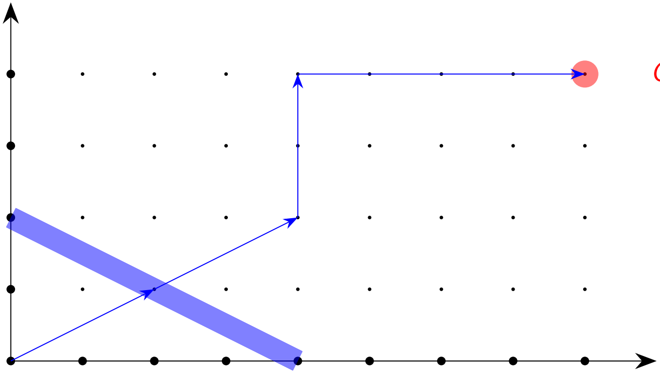

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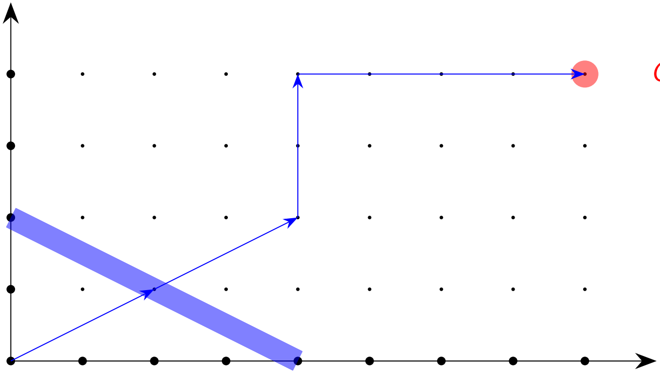

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The ConeAndPolytopeIntersection Problem

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- Polytope $P = \{x \in \mathbb{R}^N_{\geq 0} \mid A^{(P)}x = b^{(P)}\}$
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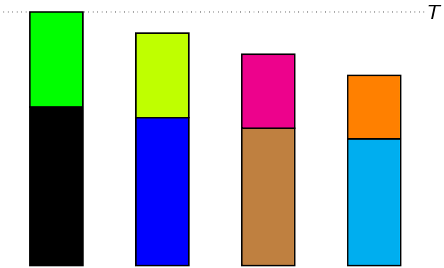

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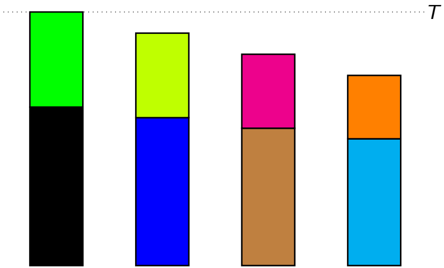

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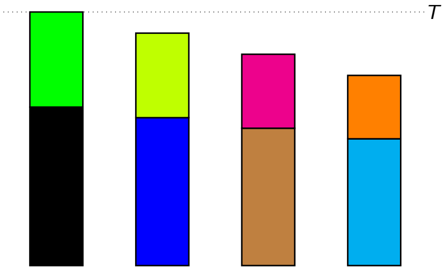

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Parameter: Number of distinct processing times d.

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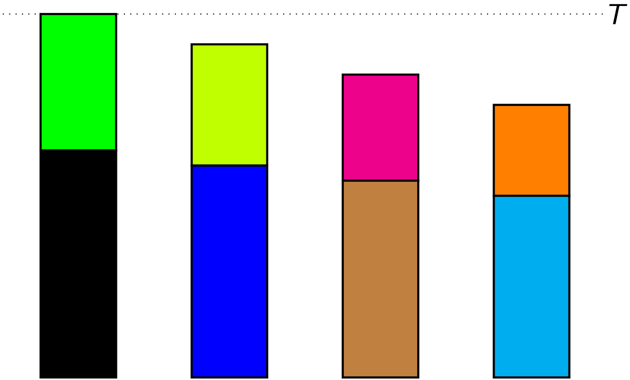

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$P||C_{\max}$-instance is feasible

$$\Longleftrightarrow$$

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Previous Results for ConeAndPolytopeIntersection

- $\langle P \rangle^{2^{O(N)}} (\langle Q \rangle \log(m))^{O(1)}$-time algorithm (Goemans & Rothvoss, 2014)

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$$\implies \log(T)^{2^{O(d)}} \langle I \rangle^{O(1)}\text{-time algorithm for } P||C_{\max}$$

Our Contribution

Theorem

ConeAndPolytopeIntersection *can be solved in time*

$$(\log(\Delta))^{2^{O(K)}} (\langle P\rangle\langle Q\rangle \log(m))^{O(1)}$$

where $K := \max\{N, M^{(P)}, M^{(Q)}\}$ *and* $\Delta = \max\{\|A^{(P)}\|_\infty, \|A^{(Q)}\|_\infty\}$.

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Theorem

$P||C_{\max}$ *can be solved in time* $(\log(p_{\max}))^{2^{O(d)}}\langle I\rangle^{O(1)} = 2^{2^{O(d)}}p_{\max}^{o(1)}\langle I\rangle^{O(1)}$.

Outline

Balancing for $P||C_{\max}$

Theorem (Govzmann et al.)

$P||C_{\max}$ has a kernel where the number of jobs of a specific type on a specific machine is bounded by $2p_{\max}$, i.e., the load of every machine is bounded by $2p_{\max}^2 d$.

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- Pre-schedule many of the jobs like in the fractional schedule.

Outline

ILP Representation

Lemma

Given $P = \{x \in \mathbb{R}^N_{\geq 0} \mid A^{(P)}x = b^{(P)}\}$ *and* $Q = \{x \in \mathbb{R}^N_{\geq 0} \mid A^{(Q)}x = b^{(Q)}\}$ *and* $m \in \mathbb{N}$, *the points* $y \in \text{int.cone}(P \cap \mathbb{Z}^N) \cap Q$ *have a one-to-one correspondence to the solutions of the following ILP, projected to block* $m+1$ *and the other blocks correspond to vectors* $y^{(1)}, \ldots, y^{(m)} \in P \cap \mathbb{Z}^N$ *such that* $y = y^{(1)} + \ldots + y^{(m)}$:

$$\begin{pmatrix} \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} \cdots \\ \cdots \end{matrix} & \begin{matrix} 0 \\ I \end{matrix} & \begin{matrix} A^{(Q)} \\ -I \end{matrix} \\ A^{(P)} & & & & \\ & A^{(P)} & & & \\ & & \ddots & & \\ & & & A^{(P)} & \\ & & & & 0 \end{pmatrix} x = \begin{pmatrix} b^{(Q)} \\ \mathbf{0} \\ b^{(P)} \\ b^{(P)} \\ \vdots \\ b^{(P)} \\ \mathbf{0} \end{pmatrix}, \quad x \in \mathbb{N}^{(m+1)N}$$

Proximity

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Theorem (Cslovjecsek et al., 2020)

For every solution x^* of the convexified relaxation, there exists a solution z^* of the corresponding integer program such that $\|x^* - z^*\|_1$ is small (and in particular does not depend on $\|b\|_\infty$).

Idea

- Compute a solution x^* of the convexified relaxation.
  - This corresponds to a "fractional solution" of the ConeAndPolytopeIntersection problem.

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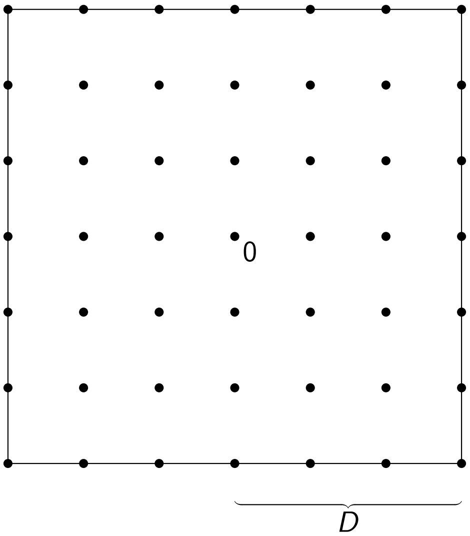
$\Longrightarrow$ We can restrict our search for a solution of the $\textsc{ConeAndPolytopeIntersection}$ problem.

$\Longrightarrow$ We only need to consider points close to x^*.

$\Longrightarrow$ We can assume that the points in P are bounded by some D (depending on the proximity bound).

Frank & Tardos, 1987



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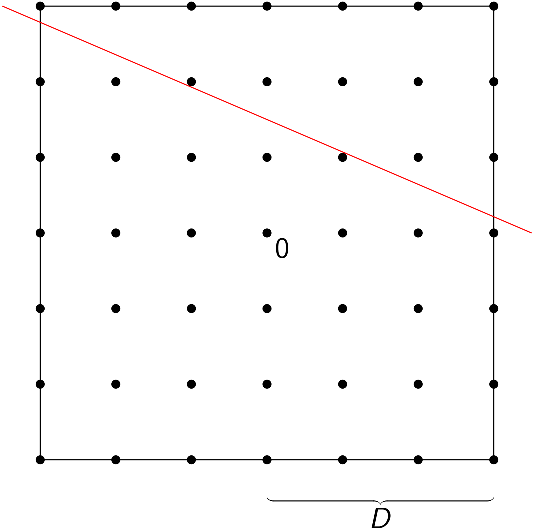

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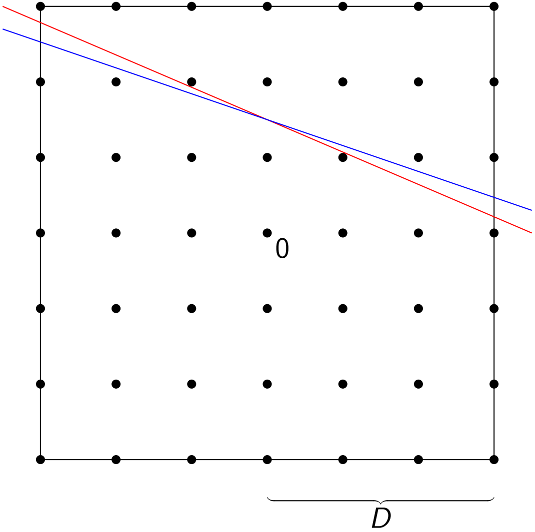

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$$\implies (\log(\Delta))^{2^{O(K)}} (\langle P \rangle \langle Q \rangle \log(m))^{O(1)} - \text{time algorithm},$$

where $K := \max\{N, M^{(P)}, M^{(Q)}\}$ and $\Delta = \max\{\left\|A^{(P)}\right\|_\infty, \left\|A^{(Q)}\right\|_\infty\}$.

Outline

1. Problem-Specific Preprocessing
2. Block-ILPs and Proximity
3. A Bound for the Vertices of the Integer Hull

Vertices of the Integer Hull

Theorem

The number of vertices of the integer hull of a polytope $P = \{x \in \mathbb{R}^N_{\geq 0} \mid Ax = b\}$ *is bounded by* $N^M O(M \log(M\Delta))^N$*, where* $A \in \mathbb{Z}^{M \times N}$ *and* $\Delta = \|A\|_\infty$*.*

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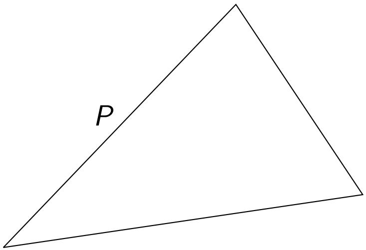
Proof is quite similar to the one of the vertex bound by Berndt et al., 2021.

Proof of the Vertex Bound



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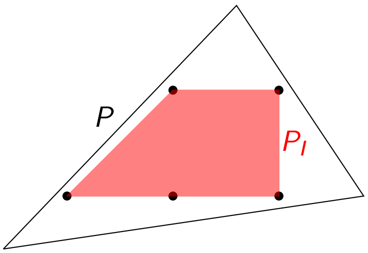

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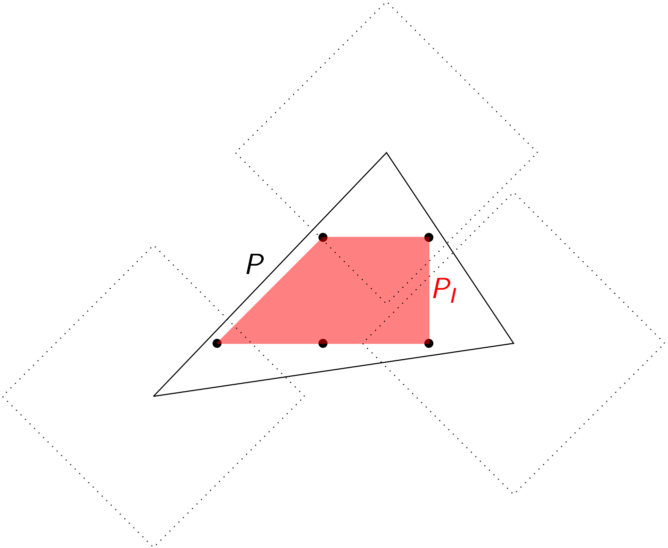

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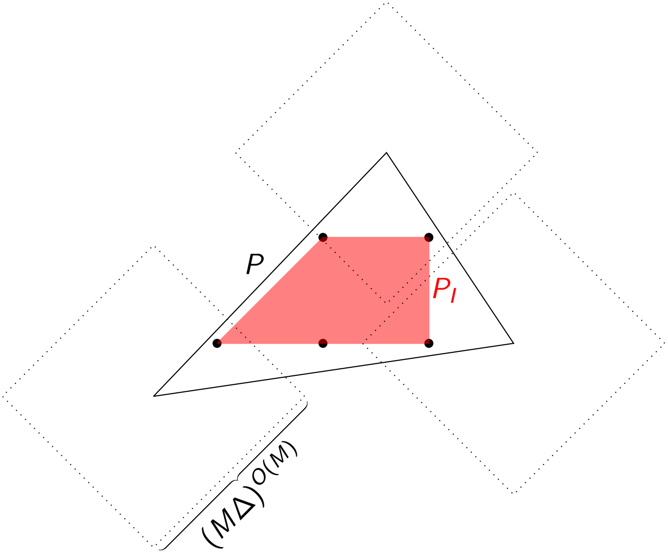

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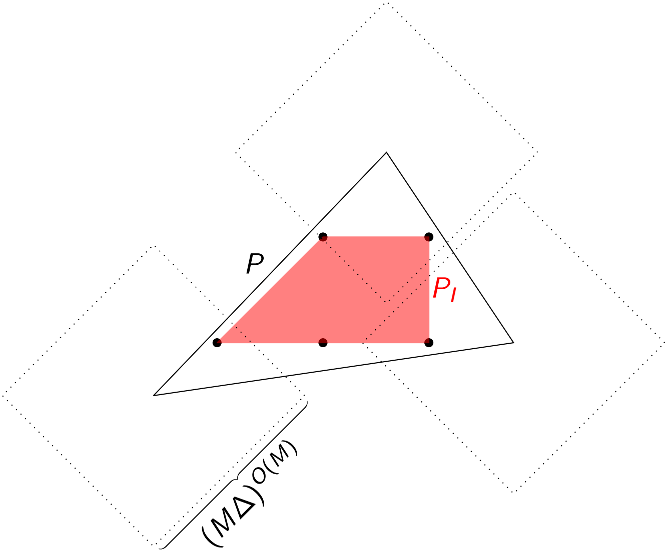

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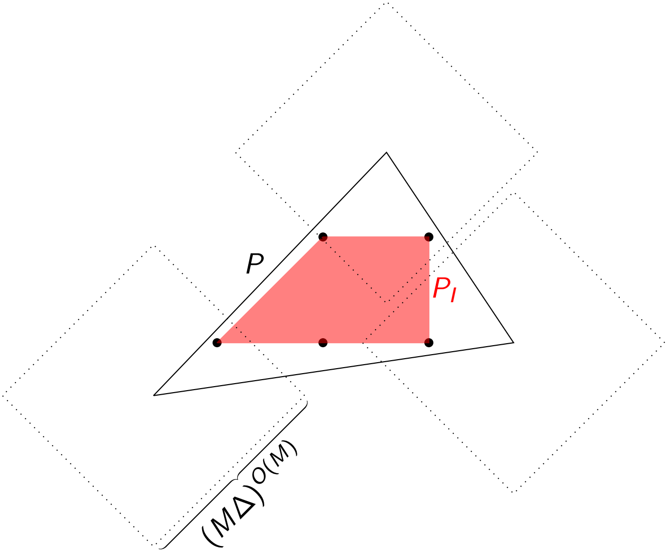

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$\leq \#$ vertices of $P \cdot \#$ grid cells

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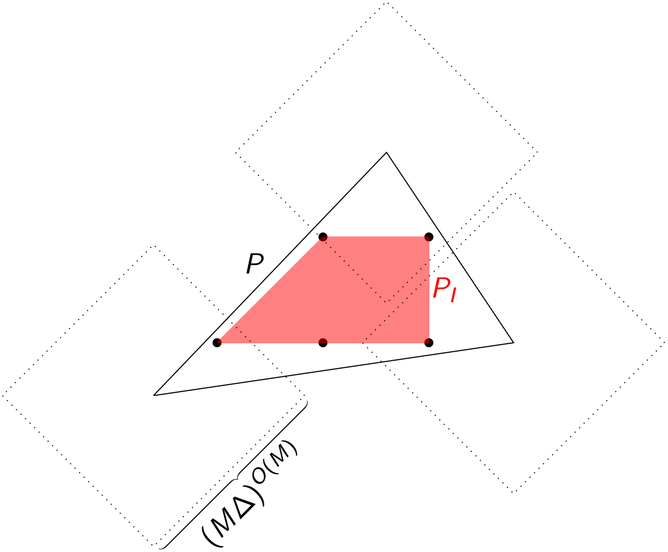

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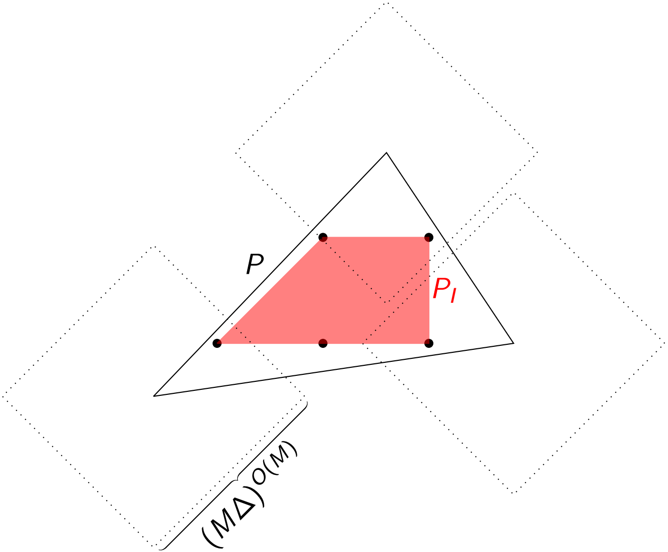

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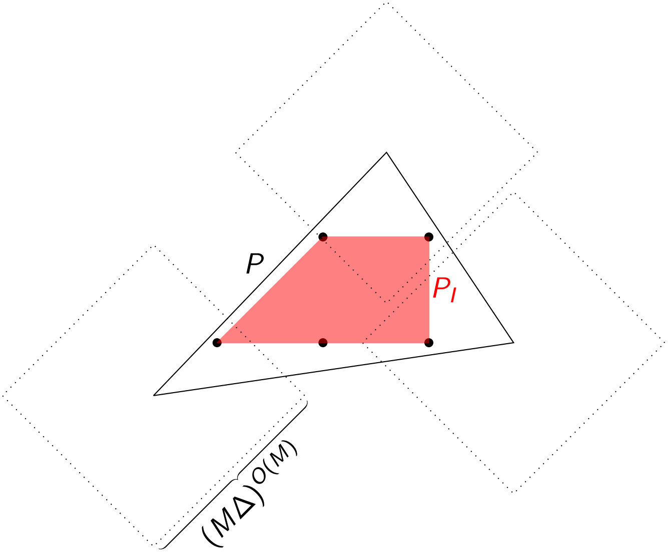

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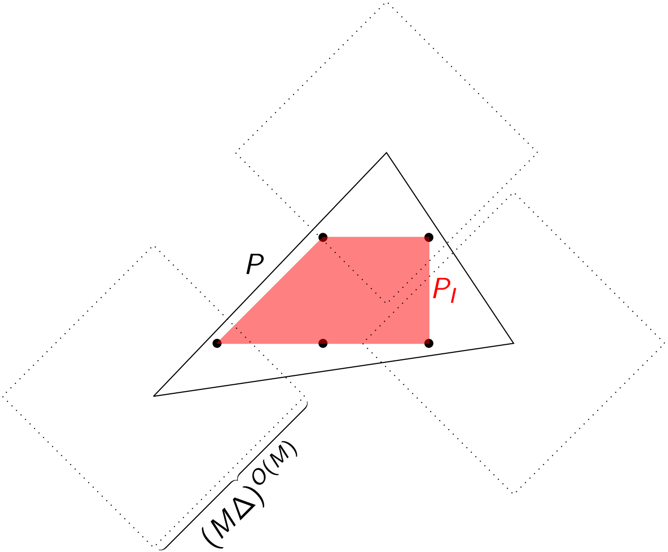

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Conclusion

Theorem

ConeAndPolytopeIntersection *can be solved in time*

$$(\log(\Delta))^{2^{O(K)}}(\langle P\rangle\langle Q\rangle\log(m))^{O(1)} = 2^{2^{O(K)}}\Delta^{o(1)}(\langle P\rangle\langle Q\rangle\log(m))^{O(1)},$$

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$P||C_{\max}$ *can be solved in time* $(\log(p_{\max}))^{2^{O(d)}}\langle I\rangle^{O(1)} = 2^{2^{O(d)}}p_{\max}^{o(1)}\langle I\rangle^{O(1)}$.

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Try applying this to your favorite scheduling problem!

Running Times

Problem	Running Time
$P\|\|\{C_{\max}, C_{\min}, C_{\text{envy}}\}$	$(\log(p_{\max}))^{2^{O(d)}} \langle I \rangle^{O(1)}$
$P\|\text{class}\|\{C_{\max}, C_{\min}, C_{\text{envy}}\}$	$(\log(\max\{\|\|n\|\|_1, p_{\max}\}))^{2^{O(d)}} \langle I \rangle^{O(1)}$
$P\|\text{cap}\|\{C_{\max}, C_{\min}, C_{\text{envy}}\}$	$(\log(p_{\max}))^{2^{O(d)}} \langle I \rangle^{O(1)}$
$P\|\text{vec}\|\{C_{\max}, C_{\min}, C_{\text{envy}}\}$	$(\log(p_{\max}))^{2^{O(d+M)}} \langle I \rangle^{O(1)}$
$P\|s_j\|\{C_{\max}, C_{\min}, C_{\text{envy}}\}$	$(\log(\max\{\|\|n\|\|_1, p_{\max}, s_{\max}\}))^{2^{O(d)}} \langle I \rangle^{O(1)}$
$P\|\|\sum w_j U_j$	$(\log(\max\{p_{\max}, w_{\max}\}))^{2^{O(d)}} \langle I \rangle^{O(1)}$
$P\|\text{vec}\|\sum w_j U_j$	$(\log(\max\{p_{\max}, w_{\max}\}))^{2^{O(d+d_{\#}M)}} \langle I \rangle^{O(1)}$
MSWBP	$(\log(\max\{p_{\max}, w_{\max} \min\{B, \|\|n\|\|_1\}\}))^{2^{O(d+m)}} \langle I \rangle^{O(1)}$
Uniform n-fold ILPs	$(\log(\max\{\|\|C\|\|_\infty, \|\|B\|\|_\infty, \|\|c\|\|_\infty\}))^{2^{O(r+s+t)}} \langle I \rangle^{O(1)}$